What Are You Implying?

Peter Wason's four card problem is one of the most celebrated probes in the study of logical thinking. The simplicity of its presentation is a siren call that leads many onto the rocks of a swift choice, later to be reconsidered. The problem may be stated as follows:

A	D	2	7

Figure 1

Each of these cards has a letter on one side and a digit on the other.

This rule may or may not be true:
if a card has a vowel on one side, then it has an even number on the other.

Which cards must you turn to check the rule for these cards?

A is almost always chosen, often in conjunction with 2, while the correct reply of A and 7 is much rarer. As I tried Wason's probe with my students I became fascinated with its possibilities and wondered how it might be extended. I was also aware that the problem is now so famous that one distinguished journal refuses to consider any more articles on the subject. I would need to find a distinctive line of approach.

I decided to retain the idea of cards, but showing statements on each side rather than digits and letters.

In the statements below, a, n and m are positive integers

1. a is even
2. a^2 is even
3. a can be written as 3n + 1
4. a can be written as 6m + 1

If we insist that a card must have distinct statements front and back, these four statements generate six cards.

1	a is even	a^2 is even
2	a is even	a can be written as 3n + 1
3	a is even	a can be written as 6m + 1
4	a^2 is even	a can be written as 3n + 1
5	a^2 is even	a can be written as 6m + 1
6	a can be written as 3n + 1	a can be written as 6m + 1

Figure 2

If we consider the logical relationships between the statements on the two faces, we find that these cards are of different types. Take Card 1 – the statement ‘a is even’ implies that ‘a^2 is even’, while the statement ‘a^2 is even’ implies ‘a is even’. If we define A I B to signify ‘A implies B’, then we might say this card is of type (I, I). What other types of card are there? We can define A RO B to signify ‘A rules out B’, and A NINRO B to mean ‘A neither implies nor rules out B’. With these definitions, the six cards fall into four types:

1 is (I, I),
3 and 5 are (RO, RO),
2 and 4 are (NINRO, NINRO), while
6 is (I, NINRO).

These are the only possible types, for if A RO B, then B RO A, in other words, (RO, I) and (RO, NINRO) represent cards that are impossible to produce. Showing A RO B is equivalent to B RO A is a good use of Modus Tollens, that is, ‘if A implies B, then (not B) implies (not A)’.

A RO B means A I (not B).
If A I (not B), then by Modus Tollens, not (not B) implies (not A).
Thus B implies (not A), that is B RO A.

Wason's problem presents students with a shock, a moment of dissonance. How best to resolve this, and to use it as a springboard for more related exploration? The common misconception over the four card problem is not trivial, for it has a generalisation that may be severely damaging to a student's mathematics, which is to claim the truth of converse theorems that are actually false. My students had a correct underlying notion (that A implies B can be reversed in some way), but overlaid with a naïve idea about how to do this. Modus Tollens represents a more ‘mature’ way to reverse A I B. Thus my task as teacher became to bring this lacuna in their thoughts out into the open, becoming something upon which they could consciously work.

I, RO and NINRO are binary relations on our set of statements. A binary relation ♥ is **reflexive** if A ♥ A for all A, **symmetric** if A ♥ B   B ♥ A for all A and B, and **transitive** if (A ♥ B and B ♥ C)   A ♥ C for all A, B and C. It appears that many people who attempt the Wason probe seem to believe in error that I is symmetric. But RO is symmetric. This suggested to me a possibly fruitful variant on Wason's problem.

You are given four cards below.
Each has a plant on one side and an animal on the other.

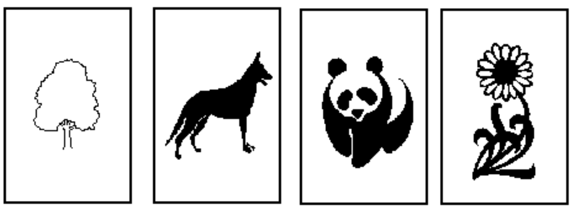

Figure 3

You are given the rule:
if one side shows a tree, the other side is not a panda.
Which cards do you need to turn over to check the rule?

This time the obvious reversal works, for ‘Tree RO Panda’ is the same as ‘Panda RO Tree’: you do need to turn over the two cards named in the question. How would my students react to this?

I chose as my subjects the group of students who had just completed their year of AS mathematics, and who were now embarking on their A2 studies. It is worth noting that the subject of 'implication' is explicit in our A Level syllabus:

To be able to construct and present a mathematical argument. Use of implies, is implied by. Use of ⇒, ⇐, ⇔ . Proof, converse, counter-example. Appropriate use of logical deduction and precise statements involving correct use of symbols and appropriate connecting language. [MEI, 2004]

I expected the groups to list the statements in pairs to find the logical relationships between them.

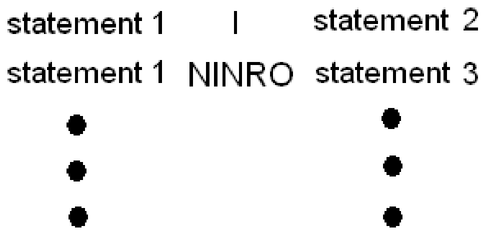

Figure 4

However, Gordon came up with the idea that we should use a grid.

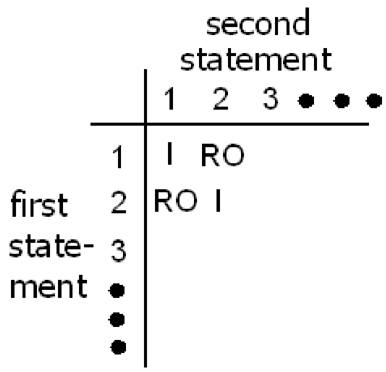

Figure 5

Another question swiftly followed:

"Can I swap my grid around?"

In other words, is this grid symmetric about the lead diagonal? Are I, RO and NINRO symmetric? Their properties can be summarised in a table below:

	Refl	Sym	Trans
I	yes	no	yes
RO	no	yes	no
NINRO	no	no	no

Figure 6

The idea of a grid certainly changed the discussion we were having, and they proved popular from then on.

My students began to use some revealing language as they discussed the cards. Gordon tried to sum up his feelings after tackling the 'statements-on-cards' problem.

"The more vague the statement, the less it implies."

Symbols were a source of concern. NINRO was given the symbol of a question mark in a circle, which pleased the class. Less pleasing to the group was the way that 'not A' had a different symbol in logic (¬ A) to that used in Probability (A'). Martin asked:

"Does that mean that probability is illogical?"

Nigel discussed how two statements were related with his partner Luke.

"So does that mean that [Statement] 1 NINROs that back again?"

NINRO swiftly became a verb for my students. Ellie summed up what she had discovered about RO.

"So if it definitely ROs one way, it's definitely going to RO the other."

Students responded well to any encouragement to use new language. "What might we call this?" I asked, referring to the relation NINRO before we had named it.

“How about A 'suggests' B?” said Luke.

“How about 'partial implication'?” offered Nigel.

"Have we ever used 'partial' elsewhere in our mathematics?" I asked. There was a brief discussion of partial fractions (and partial differentiation). It seemed that the invention and use of new language would guarantee that students would be thinking in new ways.

When I asked students at the start of the lesson to define 'A implies B', I heard this from Andrew:

"If it’s true, then it’ll work."

Later on, I heard again this use of the word 'work' for 'implies' from both Cathy and Martin:

"That only works one way.”

"It seems like a lot of them work one way but not the other way."

"It's only [Statements] 2 and 4 that work both ways."

This notion of equating 'implying' with 'working' made me think of cogs. If A turns, then B turns. But of course, if circular cogs link like this, then if B turns, then A turns. Does this go some way to accounting for the common misconception that if A implies B, then B implies A? I tried to think of other images that might analogise implication. Pipes with valves, perhaps, allowing or forbidding the flows of water in certain directions. Or the flow of currents in electrical circuits. In the end, the visual language of Venn diagrams spoke most clearly to me.

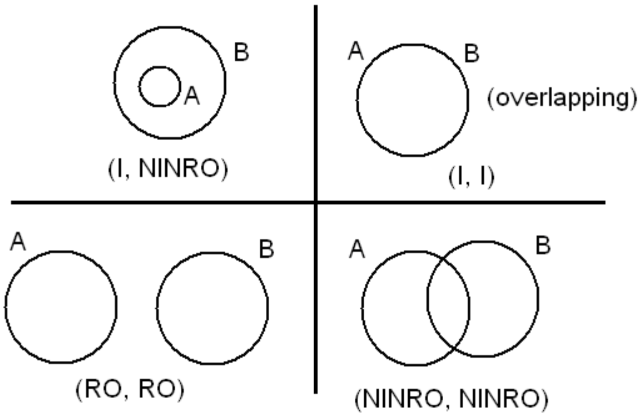

Figure 7

How would my students react to seeing this diagram in the Final Test that I had planned? Would they be able to attach pictures to ideas sensibly? In the event,

- 59% were able to correctly attach the concept to the image in the Venn diagram question.

Additionally,

- everyone was able to recall the meaning of NINRO.

- 65% of students asserted the falsity of ‘A implies B means B implies A’ and were able to give a counter-example. (Gordon wrote down the word 'example-counter' (!) - was he keen to show that there was a reversal that could validly happen?)

- 47% chose the correct panda and tree combination in the RO four card test. (13% had chosen the correct A and 7 combination in the initial Wason experiment.)

In the Final Test, I invented a new word, DONRO, standing for DOes Not Rule Out, in an attempt to see how confident my subjects now were with the idea of logical relations. I stated that DONRO is not transitive, that is, if A DONRO B and B DONRO C, it does not necessarily follow that A DONRO C, and I asked each student to pick an example to show this. The results were revealing - Ian correctly wrote this:

A: $x = 2$
B: $x^2 = 4$
C: $x = -2$

Emily wrote this:

If a shape has four sides it DONRO it is a rectangle,
if it is a rectangle it DONRO it is blue.

While this is not strictly right, it is (fairly) close to:

A: The shape S is a red quadrilateral
B: The shape S is a rectangle
C: The shape S is a blue quadrilateral

Brian, under test conditions, elegantly wrote:

A: ab is even
B: b is less than a
C: ab is odd

This is all from students who had never seen DONRO before, and who had not talked to anyone about the idea. An increased flexibility in thinking, a lack of fear in the face of new situations, and a willingness to follow intuitions, whilst at the same time realising that the ‘obvious’ may not be the truth, all of these were manifested by my students here.

Overall, I was impressed by my students' efforts to grasp these ideas, some of which I would regard as complex. They were able to assimilate (and even invent) language they had never previously encountered, and think in new ways as a result. They found the material easier than I had expected. It seems that this 'unofficial' mathematics (involving words like NINRO) offered students a fresh start, with prior achievement in 'official' mathematics implying little as to performance. The way some students did well here, much better than their examination results might have suggested they would, challenged the preconceptions that I took into these lessons. My notions of mathematical ability were also challenged. I would now need to ask, "Able? In relation to which mathematics? And in relation to which kinds of mathematics teaching?"

The logic that students employ in the mathematics classroom could hardly be a more seminal topic. To encourage students to become conscious of their own logic is to invite them to engage in meta-mathematics, an activity that could spread fruitfully into all areas of their lives. The tasks above sharpened my students' mathematical thinking and opened up ways to provide a firmer foundation for ideas of proof. Rather than seeing the Wason problem as a 'trick question', I viewed it here as a pointer to an area where important logic tends to be confused. I am now seeking out such pointers for other mathematical topics. The aim becomes to create a gentle shock, where the surprise of seeing one's mathematics exposed or confirmed is neither 'failure' nor 'success', but a growing point.

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