

Dear Editor,

*Periodic recurrence relations*

I enjoyed Jonny Griffiths' article on *Periodic recurrence relations of the type*  $x, y, y^k/x, \dots$  in *Mathematical Spectrum*, Volume 37, Number 2. A slight shift in perspective from algebra towards trigonometry enables many of the loose ends to be tied up.

The recurrence relation  $v_n = kv_{n-1} - v_{n-2}$ , with  $v_0 = 0$  and  $v_1 = 1$ , has auxiliary equation  $\lambda^2 - k\lambda + 1 = 0$  with roots  $k/2 \pm \sqrt{k^2/4 - 1}$ . Solving this in the usual way gives the following solutions.

**Case 1** If  $k = 2$  then  $v_n = n$ , and if  $k = -2$  then  $v_n = n(-1)^{n+1}$ .

**Case 2** If  $|k| > 2$  then

$$v_n = \frac{1}{\sqrt{k^2 - 4}} \left[ \left( \frac{k}{2} + \sqrt{\frac{k^2}{4} - 1} \right)^n - \left( \frac{k}{2} - \sqrt{\frac{k^2}{4} - 1} \right)^n \right].$$

(Thus, if  $|k| \geq 2$  then  $(v_n)$  is unbounded and hence divergent.)

**Case 3** If  $|k| < 2$  then  $k = 2 \cos \theta$  for some  $0 < \theta < \pi$ , and the roots of the auxiliary equation are  $\cos \theta \pm i \sin \theta$ , so that  $v_n = \sin n\theta / \sin \theta$ .

Note that the formula

$$v_n = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-1-r}{r} k^{n-2r-1}$$

quoted in Griffiths' article thus corresponds to the following trigonometric identity:

$$\sin n\theta = \sin \theta \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-r-1}{r} 2^{n-2r-1} \cos^{n-2r-1} \theta,$$

where  $\lfloor \cdot \rfloor$  denotes the integer-part function.

Certainly,  $(v_n)$  is bounded with  $|v_n| \leq \operatorname{cosec} \theta$ .

If  $v_m = 0$  then  $\theta = r\pi/m$  for some  $r$ ,  $1 \leq r \leq m-1$ , and

$$v_{m+1} = \frac{\sin((m+1)r\pi/m)}{\sin(r\pi/m)} = (-1)^r.$$

Thus, if  $r$  is even then  $(v_n)$  is periodic (with period dividing  $m$ ), and if  $r$  is odd then  $v_{2m} = 0$  and  $v_{2m+1} = 1$ , so that  $(v_n)$  has period  $2m$  (cf table 3 of Griffiths' article).

Thought of as a polynomial in  $k$ ,  $v_m(k) = 0$  has  $m-1$  distinct nonzero roots given by  $k = 2 \cos(r\pi/m)$ ,  $1 \leq r \leq m-1$ . (Since, if  $m$  divides  $m'$ , then the roots of  $v_m(k) = 0$  appear among those of  $v_{m'}(k) = 0$ , and we confirm Griffiths' observation that, as polynomials in  $k$ ,  $v_m$  divides  $v_{m'}$ , whenever  $m$  divides  $m'$ .)

The periodic sequences arise from  $k = 2 \cos(2\pi s/m)$ ,  $1 \leq s < m/2$ , and it is a standard result that, as illustrated in Griffiths' table 4, exactly  $\frac{1}{2}\phi(m)$  of them – corresponding to  $s$  and  $m$  coprime – have period  $m$ , where  $\phi(m)$  denotes Euler's totient. For example, if  $m = 15$  and  $\phi(15) = 8$  then the four entries (1.827, 1.338, -0.209, -1.956) in Griffiths' table 4 are revealed to be

$$2 \cos\left(\frac{2\pi}{15}\right), \quad 2 \cos\left(\frac{4\pi}{15}\right), \quad 2 \cos\left(\frac{8\pi}{15}\right), \quad 2 \cos\left(\frac{14\pi}{15}\right).$$

Finally, it is worth recording that the bounded sequence

$$v_n = \frac{\sin n\theta}{\sin \theta}, \quad 0 < \theta < \pi,$$

is never convergent. For the trigonometric identities

$$\begin{aligned} v_{n+1} - v_{n-1} &= 2 \cos n\theta, \\ \cos(n-1)\theta - \cos(n+1)\theta &= 2 \sin \theta \sin n\theta, \end{aligned}$$

show that, if  $v_n \rightarrow l$  then  $\cos n\theta \rightarrow 0$  and  $\sin n\theta \rightarrow 0$ , contradicting the fact that  $\sin^2 n\theta + \cos^2 n\theta = 1$ .

Yours sincerely,

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