Nomo Triples

Dr Hugh Hunt of Cambridge University presented this lovely question at Maths Jam 2010.

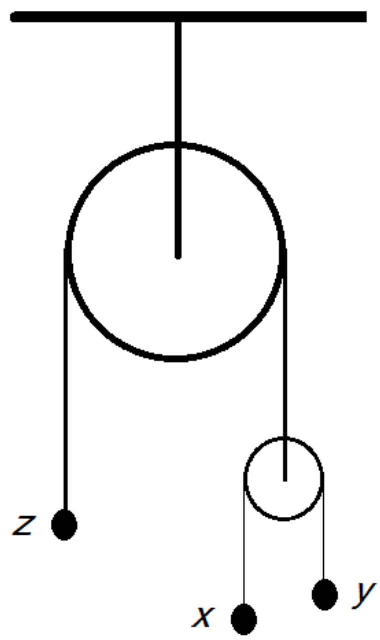

Figure 1: The 'no motion' problem

Imagine the system above released from rest with integer masses x, y and z (in kilograms), where $x < y$. Pulleys are smooth and light, strings are light and inextensible. If z remains stationary, what values are possible for (x, y, z)?

Let T be the tension in the string attached to z (in newtons), t be the tension in the string connecting x and y (in newtons), and a the acceleration of x and y (in m per s^2). Then

$yg - t = ya$ and $t - xg = xa$, so $a = g\frac{y-x}{x+y}$ and $t = xg + xg\frac{y-x}{x+y}$.

We also have $T = 2t$ and $T = zg$, so $z = \frac{2t}{g} = 2x(1 + \frac{y-x}{x+y}) = \frac{4xy}{x+y}$.

Note that T and so t and a remain constant over time. Our solutions, if you like, define a new integer triple, $(x, y, \frac{4xy}{x+y})$ with $x < y$. I've called this a Nomo Triple (hereafter called an NT), since it arises from our 'no motion' question. The ordering here is that x must be the smallest element of the triple, since $\frac{4xy}{x+y} - x = \frac{3xy-x^2}{x+y} > 0$, but $y < \frac{4xy}{x+y}$ and $y > \frac{4xy}{x+y}$ are both possible, since (3, 6, 8) and (2, 14, 7) are both NTs (there is also the special case $(k, 3k, 3k)$ where $y = \frac{4xy}{x+y}$).

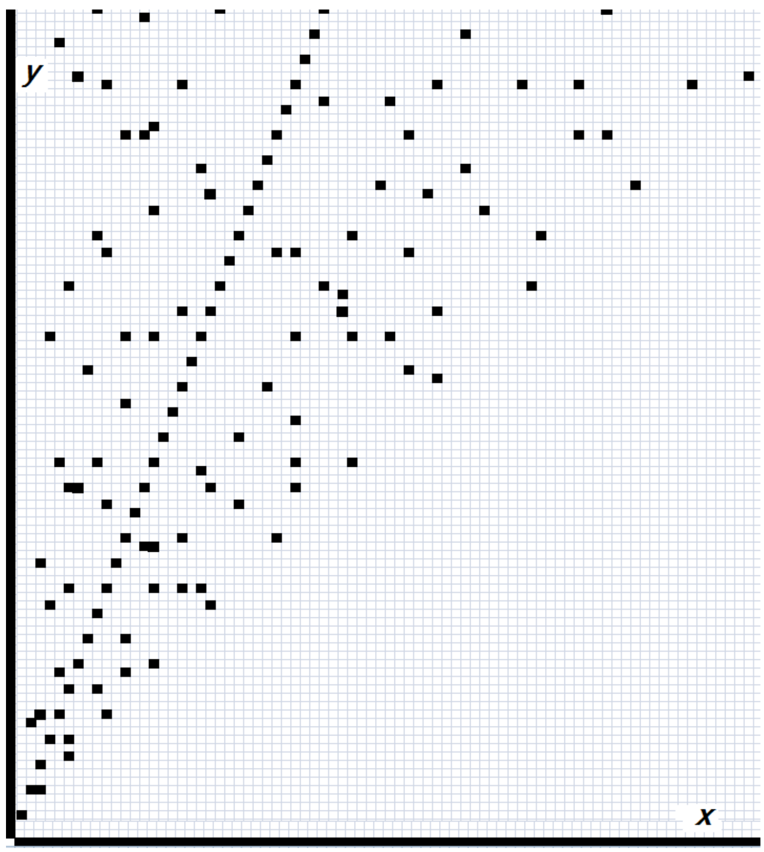

Figure 2: Early NTs

How many NTs are there? It is easy to search with a computer, and the answer is that they are not rare. The early ones are shown in Figure 2. Note that

$$(x, y, z) \text{ is an NT} \Leftrightarrow z = \tfrac{4xy}{x+y} \Leftrightarrow zk = \tfrac{4xyk^2}{kx+ky} \Leftrightarrow (kx, ky, kz) \text{ is an NT},$$

hence the NTs lying on straight lines through the origin in Figure 2. We can thus define a *primitive* NT (x, y, z) to be one where $\gcd(x, y, z) = 1$. The first few primitive NTs are given in Table 1, ordered by their sum.

x	y	z	$x+y+z$
1	3	3	7
3	6	8	17
2	14	7	23
3	15	10	28
6	10	15	31
5	20	16	41
3	33	11	47
10	15	24	49
5	45	18	68
15	21	35	71

Table 1: Early NTs ordered by sum

Can we find a parametrisation for primitive NTs, akin to that for primitive Pythagorean Triples? Finding one that yields all primitive NTs and only primitive NTs would seem to ask a lot, but one where we account for all primitive NTs, with as few as possible non-primitive NTs included, seems much more possible.

Asking Excel to chart the first few primitive NTs yields Figure 3. Careful pattern-spotting yields the following pair of parametrisations for these:

Type 1: $(\frac{n(n+2m-1)}{2}, \frac{(n+2m-1)(n+4m-2)}{2}, n(n+4m-2))$.

Type 2: $((2a-1)(2a+2b-1), (2a+2b-1)(2a+4b-1), 2(2a+4b-1)(2a-1))$

where n, m, a and b are positive integers. It is easily checked that these both always give NTs. In Figure 3 the Type 1 family is shown by filled

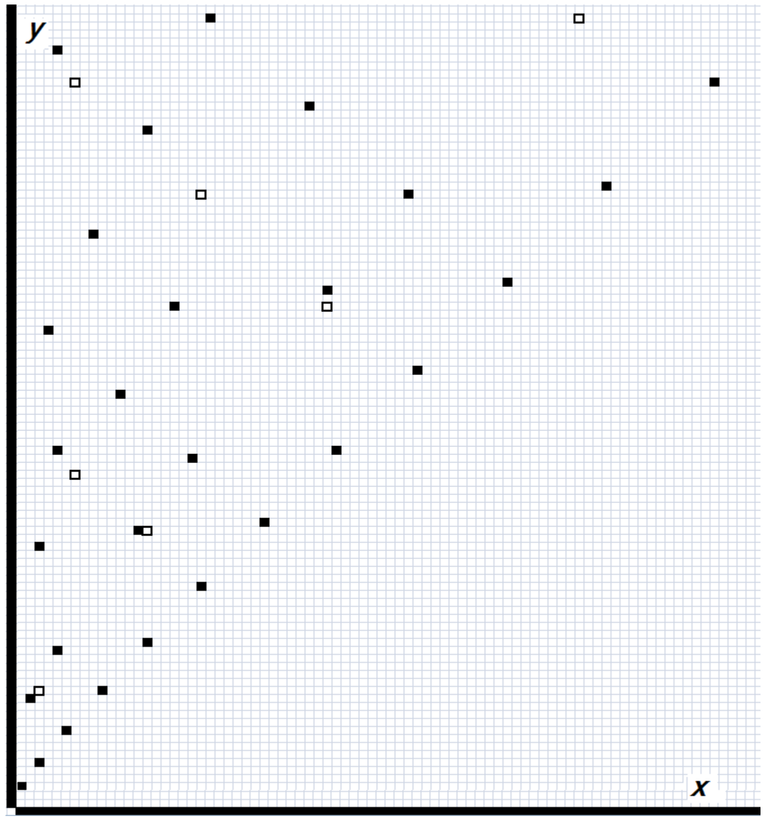

Figure 3: Early primitive NTs

squares and the Type 2 family by empty boxes. Putting $x = \frac{n(n+2m-1)}{2}$, finding m and substituting into $y = \frac{(n+2m-1)(n+4m-2)}{2}$ gives $y = \frac{4}{n^2}x^2 - x$, while putting $x = (2a-1)(2a+2b-1)$, finding b and substituting into $y = (2a+2b-1)(2a+4b-1)$ gives $y = \frac{2}{(2a-1)^2}x^2 - x$. We thus have two families of parabolas on which the primitive HTs lie. In Figure 4 the Type 1 parabola families are shown in solid lines, the Type 2 ones are shown as dashed lines, while the squares represent the small primitive NTs (which are points on the parabolas with natural number coordinates).

I conjecture that all primitive NTs are included in these parametrisations (along with some that are not primitive). A computer check for primitive NTs where x and y are less than 5 000 supports this hypothesis. The set of Type 1 NTs and the set of Type 2 NTs are disjoint, since $\frac{2}{(2a-1)^2} = \frac{4}{n^2}$ yields

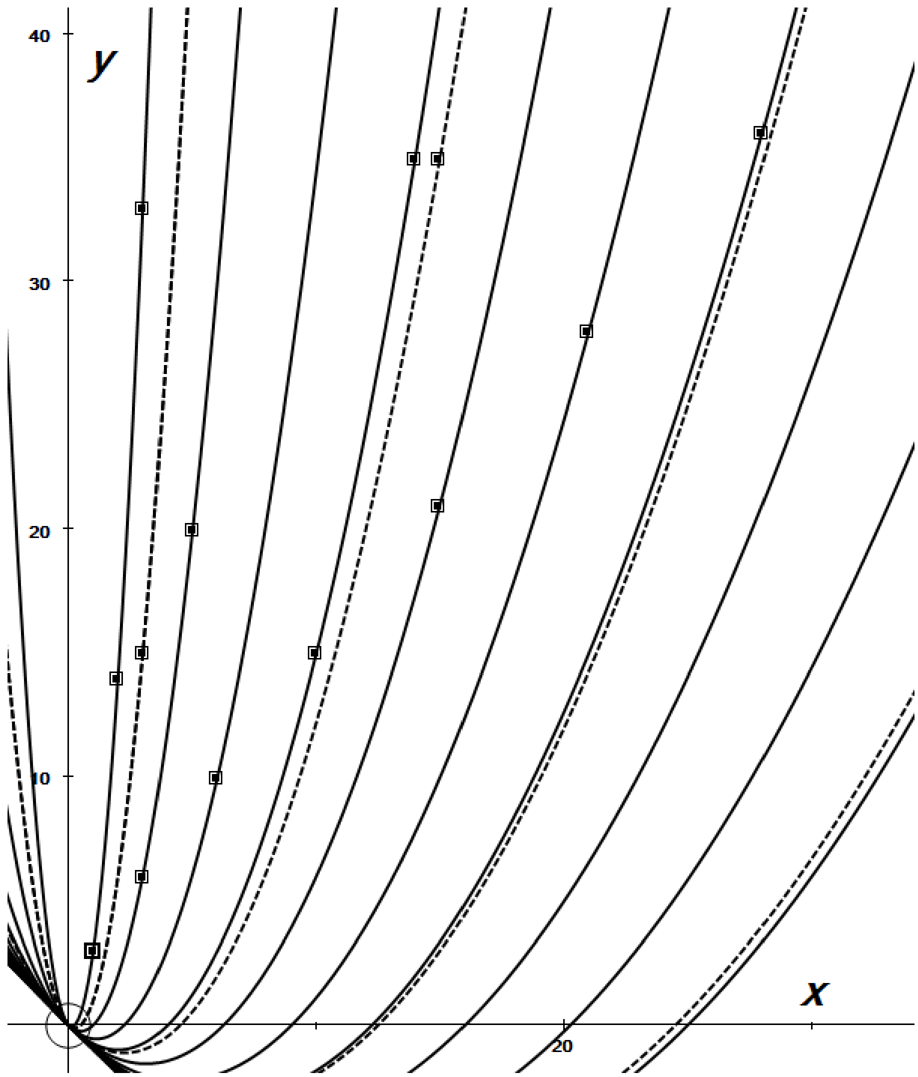

Figure 4: Type 1 and Type 2 parabolas

$n = -\sqrt{2}(2a-1)$ or $n = \sqrt{2}(2a-1)$, which is impossible.

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