Phi oh Phi oh Phi...

I wish I'd had a Φver for every time I've said to my students, "You know, Φ (The Golden Ratio, or 1.618...) crops up all over the place in maths." This is not completely surprising, as Φ will appear whenever you chance across either $x^2 - x - 1 = 0$, whose solutions are Φ and 1- Φ, or $x^2 + x -1 = 0$, whose solutions are -Φ and Φ-1. Yet you still sometimes stumble over Φ when you least expect to, and today was a case in point.

It had snowed heavily in the morning, and although the College was not closed, only two students turned up for the A2 post-lunch session, Guy and Peter. The lesson I'd planned was out of the question. I'd had half an idea for an investigation a couple of days earlier, so we set out together off-Scheme-of-Work, not knowing where we might finish. This was my question:

A triangle has a right angle, a 60º angle, and a side length 1.
How many different triangles are possible?

Innocent enough, you would have thought. Discounting reflections, there are three possible triangles, arranged below in order of size.

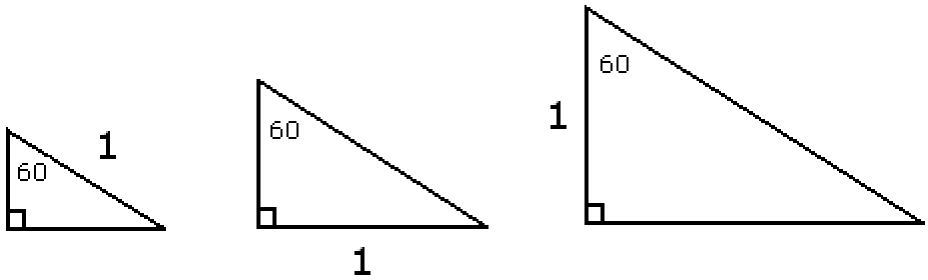

Fig. 1

Clearly the triangles are similar, and it is not hard to find the scale factors of enlargement.

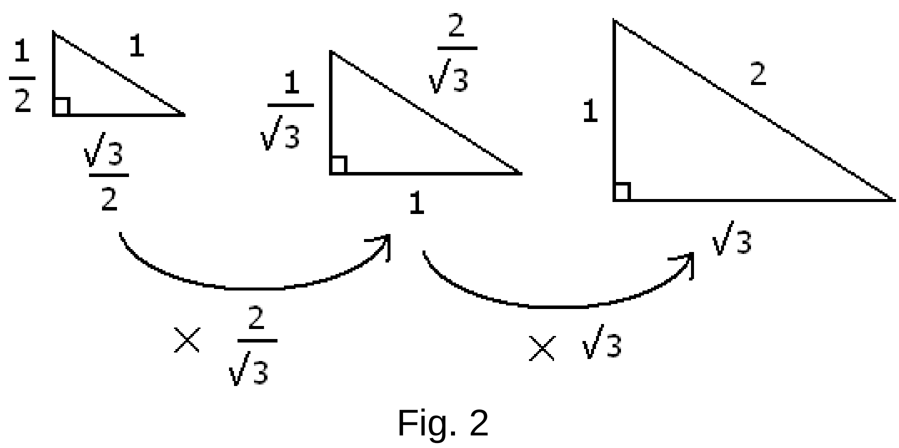

Fig. 2

I then tried to write the following:

Suppose that we change the angle that is 60º.

I was in a hurry it seemed, for I wrote 'changle' rather than 'change the angle'. This tickled my two proteges, and we agreed that whenever we wanted to change an angle in future, we would use the verb, 'to changle.' "That'll get into the dictionary one day," avowed Guy. So I wrote a second time:

Suppose that we changle 60º to α -
could the scale factor of enlargement be the same both times?

In effect, we were asking if the perimeters of the three triangles could be in geometric progression.

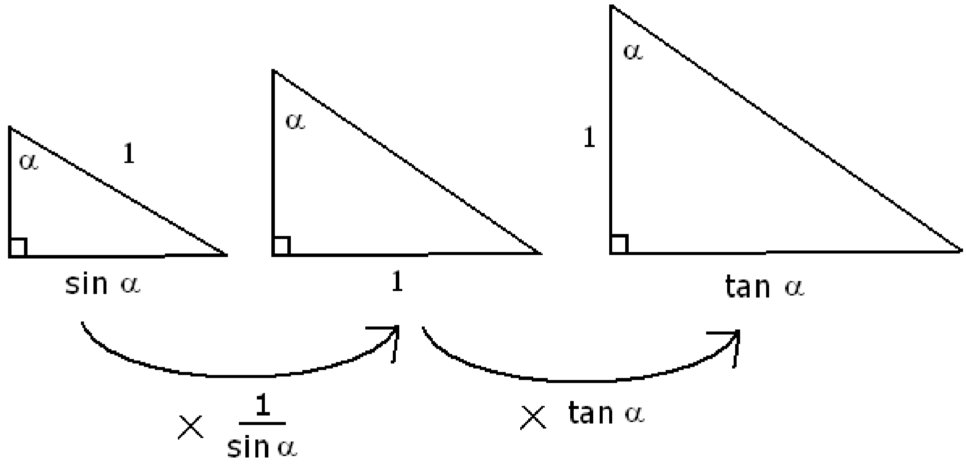

Fig. 3

We needed:

$$\frac{1}{\sin\alpha} = \frac{\sin\alpha}{\cos\alpha}$$

$$(\sin\alpha)^2 - \cos\alpha = 0$$

$$(\cos\alpha)^2 + \cos\alpha - 1 = 0$$

$$\cos\alpha = \Phi - 1, \text{ or } -\Phi.$$

Thus the angle α we seek is arccos (Φ -1), or 51.8º.

There is the assumption above that α is larger than 45º. If we consider α less than 45º, then we have this:

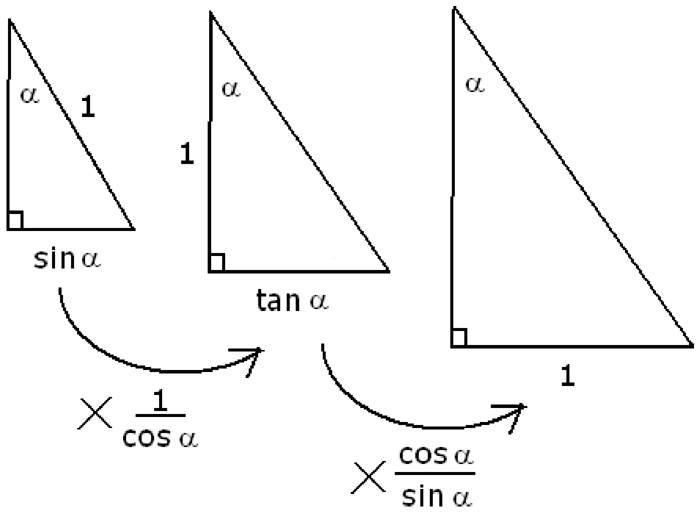

Fig. 4

So if the two scale factors are to be the same, then:

$$\frac{1}{\cos\alpha} = \frac{\cos\alpha}{\sin\alpha}$$

$$(\cos\alpha)^2 - \sin\alpha = 0$$

$$(\sin\alpha)^2 + \sin\alpha - 1 = 0$$

$$\sin\alpha = \Phi - 1, \text{ or } -\Phi.$$

Thus the angle α we seek is arcsin (Φ -1), or 38.2º.

Of course! We had in Figure 4 the same three triangles as in Figure 3, but reorganised a little. Our result told us that arcsin(Φ -1) + arccos(Φ -1) = 90º, but then a little more thought told us that arcsin x + arccos x = 90º (1) was in fact an identity, a simple consequence of Figure 5.

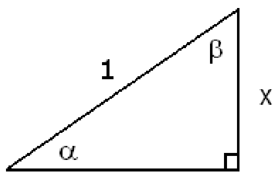

Fig. 5

The three of us knew that sin x = cos (90º - x), but we couldn't remember ever seeing this written quite as (1). The familiar in a unfamiliar guise...

Another question occurred to us:

If we change 60º to β, is it possible for the <u>areas</u> of the three triangles to be in geometric progression?

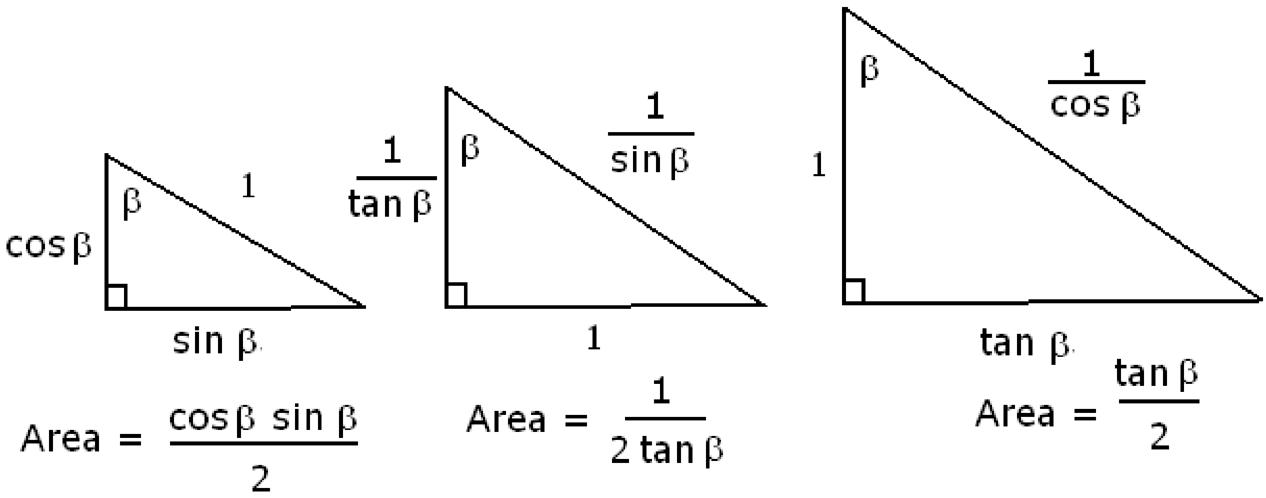

Fig. 6

So $\dfrac{\cos\beta \times \sin\beta}{2} * \dfrac{\sin\beta}{2\cos\beta} = \dfrac{(\cos\beta)^2}{4(\sin\alpha)^2}$

So $(\sin\beta)^4 = (\cos\beta)^2$

$(\sin\beta)^4 + (\sin\beta)^2 - 1 = 0$

"There's that equation again," said Guy.

$(\sin\beta)^2 = \Phi - 1$, or $-\Phi$.

Thus the angle β we seek is arcsin √(Φ -1), or 51.8º.

Surprise! [1] Now that figure of 51.8º has appeared before! Is this another example of a simple trig identity in an unfamiliar form? We have here that:

arcsin √(Φ -1) = arccos (Φ -1)

Is this generally true? Does arcsin √x = arccos x for a range of possible x?

Drawing the graph of y = arcsin √x - arccos x, we found that y = 0 for just one value of x, namely Φ -1. So most definitely we do not have an identity here.

Taking β as less than 45º in the above, we found a value for β of arccos √ (Φ -1), or 38.2º. So the solution for the equation arccos √x = arcsin x is also Φ -1. Figure 7 shows the graphs of y = arcsin √x - arccos x, and y = arccos √x - arcsin x, meeting on the x-axis at Φ -1.

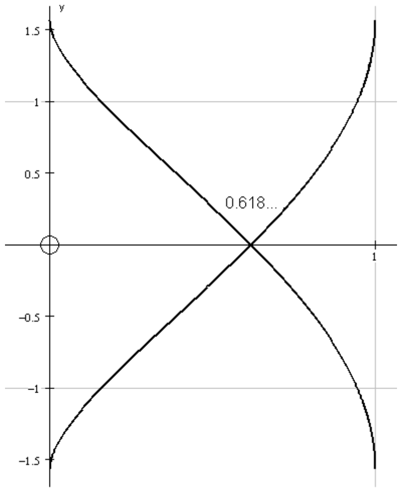

Fig. 7

Thinking about it afterwards, I could see why this happens.

If arccos √x = arcsin x = *β*, then by Pythagoras:

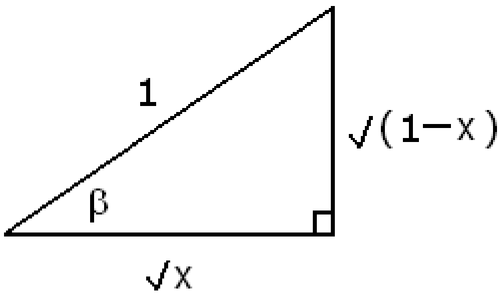

Fig. 8

So sin *β* = x = $\frac{\sqrt{(1-x)}}{1}$, which gives $x^2 + x - 1 = 0$.

"There's that equation again!"

With thanks to Guy Collins and Peter Temple

Footnote: Perhaps this makes sense in retrospect. If three similar triangles have their perimeters in geometric progression, then corresponding sides will be a, ka and k^2a, so their areas will be as a^2 to k^2a^2 to k^4a^2, in other words, the areas will be in geometric progression too.

Jonny Griffiths, 4th March 2005, Paston College

www.jonny-griffiths.net