Periodic Recurrence Relations of the type x, y, y^k/x…

We are all likely to remember the first time we were struck by the way that some mathematical activity "came back to where it started", sometimes quite unexpectedly.

Maybe we observed that if you start with 3, and take it away from 7, you get 4. Now take 4 away from 7 again, and you get back to 3. Does this always work? A stronger motivation for embarking on algebra it is hard to imagine:

$$x, a-x, a-(a-x) = x$$

You can if you wish regard this as a recurrence relation, or RR, period 2, where $u_n = a - u_{n-1}$, and where u_0 can be anything.

Can we find any other RRs that do this? Perhaps the simplest is x, a/x, x,... (period 2). If you wonder what happens if we try to make this more general, and also if we can get away from period 2, we could try:

$$x, (ax + b)/(cx+d),...$$

For period 2, doing the algebra implies the RR must be of the form (ax+b)/(cx-a) to be non-trivial, which includes both of the RRs above as special cases.

What about period three? The algebra gets a little more complicated, but eventually gives:

$x, (ax-(a^2 + ab + b^2))/(x + b), (bx + (a^2 + ab + b^2))/(-x + a), x,...$ as the non-trivial solution.

This idea can be continued for higher periods, although the algebra gets pretty dense!

What happens if you extend this idea to RRs that include more than just the previous term? In other words, instead of looking at x, f(x), f(f(x))..., we look at x, y, f(x,y), f(y, f(x,y)),... . Can this be periodic?

One of the happiest memories of my years at school is of being shown the following:

$$x, y, (y+1)/x, (1+x+y)/xy, (x+1)/y, x, y,... .$$

The simplicity of the function, and the way in which the algebra organised itself to return to its starting point, struck me as marvellous.

I suppose that, ever since, I have been on the lookout for RRs that return to their starting points. So it wasn't surprising that the other day I found myself looking at these:

x, y, 1/(xy), x, y, ...(period 3), and x, y, y/x, 1/x, 1/y, x/y, x, y,...(period 6).

Could I get a period four? A period five? Suddenly, it seemed a fruitful idea to look at x, y, x^jy^k... and to see what that would yield. (Actually, looking at x, y, jx+ky... gives virtually the same maths.)

In effect, I was producing two more RRs, u_n and v_n, where the nth term in the original RR would be $x^{un}y^{vn}$. So the sequence for u_n and v_n is given in Table 1:

n	un	vn
0	1	0
1	0	1
2	j	k
3	jk	k^2+j
4	$j^2 + jk^2$	k^3+2jk
5	$2j^2k+jk^3$	$k^4 + 3jk^2+ j^2$
6	$j^3 + 3j^2k^2 + jk^4$	$k^5+4jk^3 + 3j^2k$
7	$3j^3k+4j^2k^3+jk^5$	$k^6+5jk^4+6j^2k^2+j^3$

Table 1

So, if I take j = -1, k = -1, then $u_3 = 1$, $v_3 = 0$, $u_4 = 0$, $v_4 = 1$, I get my 1/xy RR, with period three.

So is period four possible? This would mean that $j^2 + jk^2 = 1$ and $k^3 + 2jk = 0$, which gives us:

$$(j, k) = (1,0), \text{ or } (-1,0), \text{ or } (i, 1- i), \text{ or } (i, i - 1), \text{ or } (- i, 1 + i), \text{ or } (- i, - i - 1).$$

So if we are restricting ourselves to real solutions, x, y, x^{-1}, y^{-1}, x, y,… is the only possibility for period four with this kind of RR, it seems.

(Let me note that x, y, (x + a)(y + a)/(-y),… is periodic, period 4.)

How about period five? This would mean that:

$$2j^2k + jk^3 = 1 \text{ and } k^4 + 3jk^2 + j^2 = 0.$$

These give you, after lots of substitution, the equation: $k^{10} + 11k^5 - 1 = 0$, which is a quadratic in k^5.

Solving this gives $k^5 = (-11 \pm 5\sqrt{5})/2$.

What do you get if you take the fifth root here? To my delight, I got the following:

$k = \phi - 1$ or $- \phi$, where ϕ is the Golden ratio $(1+\sqrt{5})/2$, and in both cases j = -1.

This suggests that the RR x, y, $1/(xy^{\phi})$ may have period five, and it does: it is a very enjoyable exercise to check this through, simply using $\phi^2 = \phi + 1$. Maybe a candidate for the most beautiful RR ever?

It seems that j is very often -1 with the maths generated here. Let's simplify things by restricting ourselves to j as -1 for the moment. So we are now just looking at RRs of the form: x, y, y^k/x,… . Our table simplifies to Table 2:

n	un	vn
0	1	0
1	0	1
2	(-1)	k
3	(-k)	k^2-1
4	$1-k^2$	k^3-2k
5	$2k-k^3$	k^4-3k^2+1
6	$3k^2-1-k^4$	k^5-4k^3+3k
7	$4k^3-3k-k^5$	$k^6-5k^4+6k^2-1$

Table 2

So we can see that the u_n and the v_n here are almost the same sequence, just out by a minus sign and one term. So:

$$u_n = - v_{n-1}.$$

A graphics calculator and a short program becomes very useful here to check out what happens. Enter say, 2 for x and 3 for y, then a value for k, and you can see what happens as the RR progresses. You can see if you do this that, if $-2 < k < 2$, then the terms remain bounded, whereas if k is outside this range, then the terms head off to infinity. It appears that our only chance of periodic behaviour lies in the range $-2 < k < 2$. Why should this be? (Curiously, the condition for a point to be in the Mandelbrot set is very similar.)

We can easily get a simple RR for the v_n, that is: $v_n = kv_{n-1} - v_{n-2}.$

So can we solve this RR? We can, and it looks like this: $v_0 = 0$, $v_1 = 1$, and

$v_n = ({}^{n-1}C_0)k^{(n-1)} - ({}^{n-2}C_1)k^{(n-3)} + \ldots + (-1)^r({}^{n-1-r}C_r)k^{(n-2r-1)} + \ldots$ (ending in the term in k^1 or k^0).

Checking this formula (even if the check is rather rough at the edges!), we have:

$$kv_{n-1} - v_{n-2} = \Sigma\, (-1)^r k^{(n-2r-1)}[\, ({}^{n-2-r}C_r) + ({}^{n-2-r}C_{r-1})] = \Sigma\, (-1)^r k^{(n-2r-1)}[\, ({}^{n-1-r}C_r)] = v_n.$$

So for $n = 2t$, $v_{2t} = ({}^{2t-1}C_0)k^{(2t-1)} - ({}^{2t-2}C_1)k^{(2t-3)} + \ldots + (-1)^r({}^{2t-1-r}C_r)k^{(2t-2r-1)} + \ldots + (-1)^{t-1}({}^{t}C_{t-1})k^1.$

For $n = 2t+1$, $v_{2t+1} = ({}^{2t}C_0)k^{(2t)} - ({}^{2t-1}C_1)k^{(2t-2)} + \ldots + (-1)^r({}^{2t-r}C_r)k^{(2t-2r)} + \ldots + (-1)^{t-1}({}^{t-1}C_{t-1})k^0.$

For a periodic RR, $v_n = 0$ for some $n > 0$. So we end up trying to solve equations like:

$$({}^8C_0)k^8 - ({}^7C_1)k^6 + ({}^6C_2)k^4 - ({}^5C_3)k^2 + ({}^4C_4) = 0$$

or

$$({}^7C_0)k^7 - ({}^6C_1)k^5 + ({}^5C_2)k^3 - ({}^4C_3)k = 0.$$

The coefficients here lie in diagonal lines within Pascal's Triangle (Table 3):

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                        1            v3
                    1       1              v5
                1       2       1                 v7
            1       3       3       1
        1       4       6       4       1
    1       5       10      10      5       1
1       6       15      20      15      6       1
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Table 3

Does this kind of equation have real solutions for k between -2 and 2? The possibly surprising news is these equations seem remarkably keen to give such values.

For example, $v_{12} = 0$ gives the following equation: $k^{11} - 10k^9 + 36k^7 - 56k^5 + 35k^3 - 6k = 0$.

This has 11 real roots, 0, ±1, ±√2, ±√3, ±√(2+√3), ±√(2-√3), which are all between -2 and 2. Using our graphics calculator, we can see what period each of these values of k gives: -1 gives period 3, 0 period 4, 1 gives period 6, ±√2 period 8, ±√3 period 12, and ±√(2±√3) period 24.

Carrying this out for all small values of n we get Table 4:

Equation	No of Solutions	Periods
v2=0	1	4
v3=0	2	3,6
v4=0	3	4,8,8
v5=0	4	5,5,10,10
v6=0	5	3,4,6,12,12
v7=0	6	7,7,7,14,14,14
v8=0	7	4,8,8,16,16,16,16
v9=0	8	3,6,9,9,9,18,18,18
v10=0	9	4,5,5,10,10,20,20,20,20
v11=0	10	11,11,11,11,11,22,22,22,22,22

Table 4

What does this suggest? For starters, that whenever $n \geq 3$, it is possible to find a value of k that gives period n, which is very pleasing! So that if we seek a value of k so that x, y, y^k/x... has period 2049, then such a k exists, it seems.

It also seems that if $p = 2n+1$ is prime then there are n values of k that give x, y, y^k/x as periodic period p, and a further n that give period 2p.

How many different values of k are there for each period? See Table 5:

Period	Number of solutions for k	Solutions for k
3	1	(-1)
4	1	0
5	2	(-1.618), 0.618
6	1	1
7	3	(-1.802), (-0.445), 1.247
8	2	(-1.414), 1.414
9	3	(-1.877), 0.357, 1.520
10	2	(-0.618), 1.618
11	5	(-1.919),(-1.310),(-0.285), 0.831, 1.683
12	2	(-1.732), 1.732
13	6	(-1.942),(-1.497), (-0.709), 0.241, 1.136, 1.770
14	3	(-1.247), 0.445, 1.802
15	4	(-1.956), (-0.209), 1.338, 1.827
16	4	(-1.848), (-0.765), 0.765, 1.848
17	8	***
18	3	*** , 1.877
19	9	***
20	4	*** , 1.902

Table 5

It seems that, if x, y, y^k/x is periodic, then x, y, y^{-k}/x is too, yet not always with the same period (there may be a halving or a doubling). So x, y, $y^{\sqrt{2}}/x$ gives period 8, and so does x, y, $y^{-\sqrt{2}}/x$. However, x, y, y^{ϕ}/x...has period 10, while x, y, $y^{-\phi}/x$...has period 5.

Also, it appears that the top solution for k is steadily increasing. This gives us the conjecture that there is a sequence k_3, k_4, k_5...with $k_n < k_{n+1} < 2$ for all n such that x, y, y^{kn}/x is periodic, period n. (With $k_3 = -1$, $k_4 = 0$, $k_5 = \phi - 1$, $k_6 = 1$, $k_8 = \sqrt{2}$, $k_{10} = \phi$, and $k_{12} = \sqrt{3}$)

So it seems that finding a value of k that gives $v_n = 0$ means that we must have found something periodic. Our experimenting suggests that, if $v_n = 0$, then u_n must be either 1 or -1, and that v_{n+1} is always 1 or -1. The 'pressure', if I may call it that, for a RR to go on to y once it has returned to x is enormous, it appears. Why should this be?

A little thought suggests that, if k is a root of $v_n = 0$, then k will be a root of $v_{2n} = 0$, indeed of $v_{an} = 0$. So if, as our experimenting suggests, v_m always has m-1 distinct roots, we have that, if $n \mid m$, then $v_n \mid v_m$.

Take v_4 and v_8: is it immediately obvious that $(k^3 - 2k)$ divides $(k^7 - 6k^5 + 10k^3 - 4k)$ exactly? These polynomials clearly have some interesting properties. The quotient here turns out to be $k^4 - 4k^2 + 2$: what is the significance of this?

One last thought: a brief proof that if $|k| \geq 2$, then the v_n diverge.

We have $v_0 = 0$, and $v_1 = 1$, and we have that $v_n = k\, v_{n-1} - v_{n-2}$

Suppose $|k| \geq 2$. Certainly, $|v_1| \geq |v_0| + 1$ Suppose $|v_n| \geq |v_{n-1}| + 1$

$$
\begin{aligned}
|v_{n+1}| &= |kv_n - v_{n-1}| \\
&\geq \big|\,|k|\,|v_n| - |v_{n-1}|\,\big| \text{ (backwards triangle inequality)} \\
&= |k|\,|v_n| - |v_{n-1}| \\
&= |v_n| + (|k|-1)|v_n| - |v_{n-1}| \\
&\geq |v_n| + (|k|-1)(|v_{n-1}|+1) - |v_{n-1}| \\
&= |v_n| + (|k|-2)|v_{n-1}| + |k| - 1 \\
&\geq |v_n| + 1.
\end{aligned}
$$

So, by induction, $|v_n| \geq |v_{n-1}| + 1$ for all n.
Hence $|v_n| \geq n$ for all $n \geq 0$, and the v_n diverge.

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