Totally Perfect Siders: Jonny Griffiths

"Maths is all around us," we tell our students, and sometimes that mathematics impresses itself forcibly upon us. Last week I placed my order in a coffee shop, feeling happy to have a moment to sit down. It was not long before I was struck by a pleasing pattern on the floor.

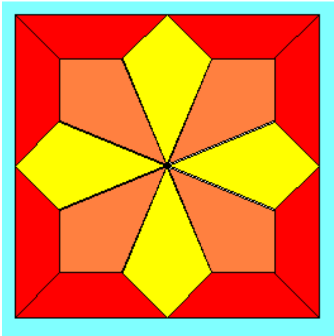

Figure 1

"Looks Islamic," I thought to myself as I worked out angles. I found myself absorbed to the exclusion of everything else: I was having a minor Archimedes moment. (Archimedes was wrapped up in mathematics as enemy soldiers burst in to murder him, so the story goes.) Happily, rather than being disturbed by warriors about to take my head off, my reverie was ended by the arrival of my bacon and Brie pitta. Later that day, I tried to explore that central tile a little more.

"The angles are 45°, 90° and 112.5°. There's the standard kite tiling..."

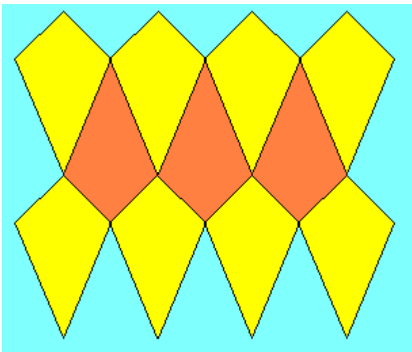

Figure 2

"There's this, using an extra rhombus tile..."

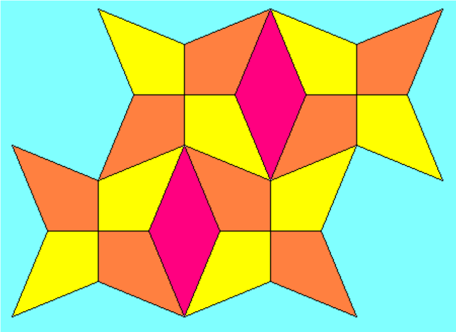

Figure 3

I found myself arranging several tiles as follows:

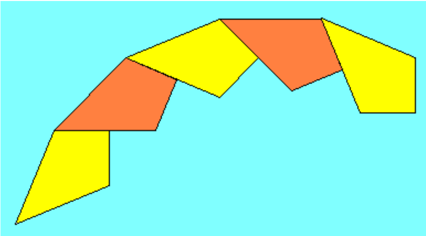

Figure 4

This would extend to give a regular 16-sided polygon. But then there was also this:

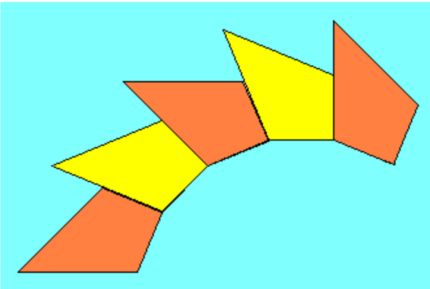

Figure 5

Also giving a regular 16-sided polygon. "But as an Outsider rather than as an Insider," I reflected.

A question started to form in my mind: "Could an n-sided polygon give n different regular polygons in this way, as either an Outsider or an Insider?"

Starting with a triangle seemed to be a good idea.

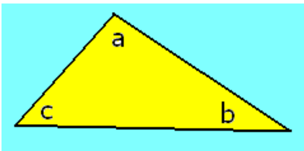

Figure 6

I would need:

$$a + b + c = 180,\ a + b = 180 - \frac{360}{n_1},\ b + c = 180 - \frac{360}{n_2},\ \text{and } c + a = 180 - \frac{360}{n_3}.$$

Adding these last three, I get $\frac{1}{2} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$, for instance, $\frac{1}{2} = \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$.

I began to connect with Egyptian mathematicians and their system of fractions, built solely from unit ones (fractions with a 1 on the top, as we would say). I thought too of resistors in parallel, where a 4-ohm and a 5-ohm and a 20-ohm together produce a 2 ohm one. (In Figure 7, if R is the resistance of the whole component, then $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.)

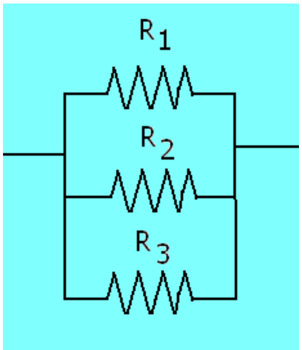

Figure 7

Using these numbers, 4, 5 and 20, for a possible triangle meant solving a + b = 90, b + c = 108 and c + a = 162, which gave a = 72°, b = 18°, c = 90°. This was the result:

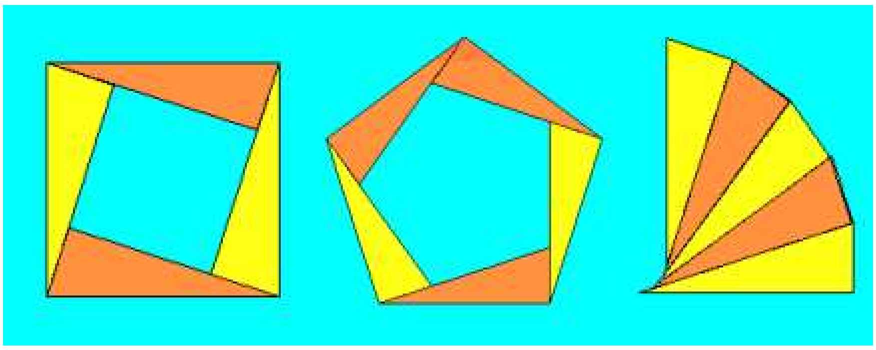

Figure 8

The triangle acts as an Insider in three different ways (a triangle can never be an Outsider.) A neat way of linking the topic of resistors-in-parallel to geometry?

I felt in need of a name for these polygons. Outsiders, Insiders... "How about, Siders?" I wondered. Side-by-siders, even. Google makes it blessedly easy to check on whether a possible name has helpful or unhelpful connotations. ('Polytiles' was rejected quite quickly as a name.) Might there be Fool's Sider, that could be called a con-sider? I thought back to the death of Archimedes, the King of Mathematics, and 'regisiders' offered some kind of fanciful confirmation.

Moving on to quadrilaterals, I needed to solve:

$$a + b = 180 \pm \frac{360}{n_1},\ b + c = 180 \pm \frac{360}{n_2},\ c + d = 180 \pm \frac{360}{n_3},\text{ and } d + a = 180 \pm \frac{360}{n_4}.$$

Adding the first and third equations means that n_1 and n_3 must be of opposite polarity but the same size. (Adding the second and fourth shows the same is true for n_2 and n_4.) So it seems that only two different polygons are possible with a quadrilateral, but it seems too that n_1 and n_2 can be anything: you will get one Outsider and one Insider pattern for n_1, and similarly for n_2. For example, taking $n_1 = 5$, and $n_2 = 7$:

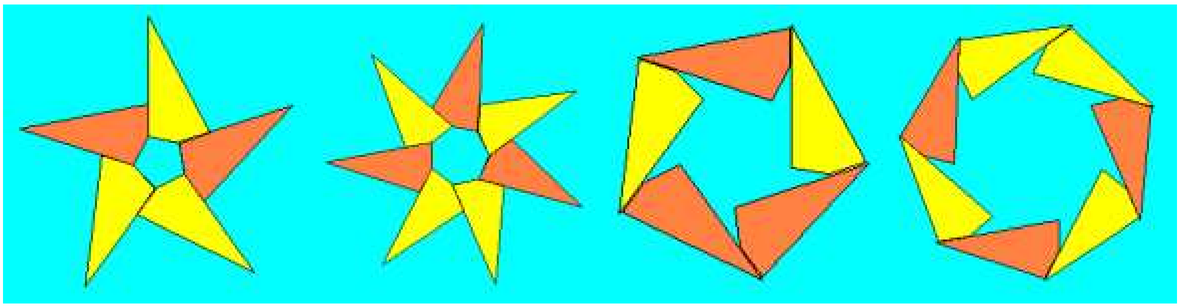

Figure 9

One other point: we get some freedom in choosing the angles here. Solving the equations for the example above, we get the four angles to be:

a, 231.4 - a, 20.6 + a, and 108 - a.

So we can choose a, within limits: what would be a helpful choice? Choosing a so that the quadrilateral tiles? But all quadrilaterals tile. "Could we get opposite angles to add to something? Say a + c = b + d = 180..."

This gives a value for a of 79.7°, and means that the quadrilaterals can form a straight line, a fact that can be used to generate the following rhombus as a kind of 'bonus shape'.

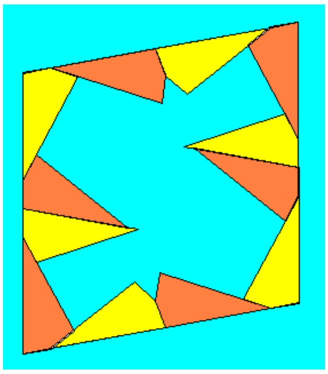

Figure 10

I felt the time had come to tackle the m-sided shape in general. The equations are:

$$a_1 + a_2 = 180 \pm \frac{360}{n_1},\ a_2 + a_3 = 180 \pm \frac{360}{n_2},\ \ldots\quad a_m + a_1 = 180 \pm \frac{360}{n_m}$$

$$\text{Adding, } 2(m-2)\times 180 = 180m + 360\left(\pm\frac{1}{n_1}\ldots\pm\frac{1}{n_m}\right)$$

$$\Rightarrow 2m - 4 = m + 2\left(\pm\frac{1}{n_1}\ldots\pm\frac{1}{n_m}\right) \Rightarrow \frac{m-4}{2} = \left(\pm\frac{1}{n_1}\ldots\pm\frac{1}{n_m}\right)$$

It is easy to check that if I insist that n_1 to n_m are different, and are integers greater than 2, I cannot have m greater than 6. So if m = 5?

$$\text{I need } \frac{1}{2} = \left(\pm\frac{1}{n_1}\ldots\pm\frac{1}{n_5}\right) : \text{how about } \frac{1}{2} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{10} - \frac{1}{60}\ ?$$

This gives a + b = 270, b + c = 252, c + d = 240, d + e = 144, and e + a = 174,

which solve to give a = 144°, b = 126°, c = 126°, d = 114°, and e = 30°.

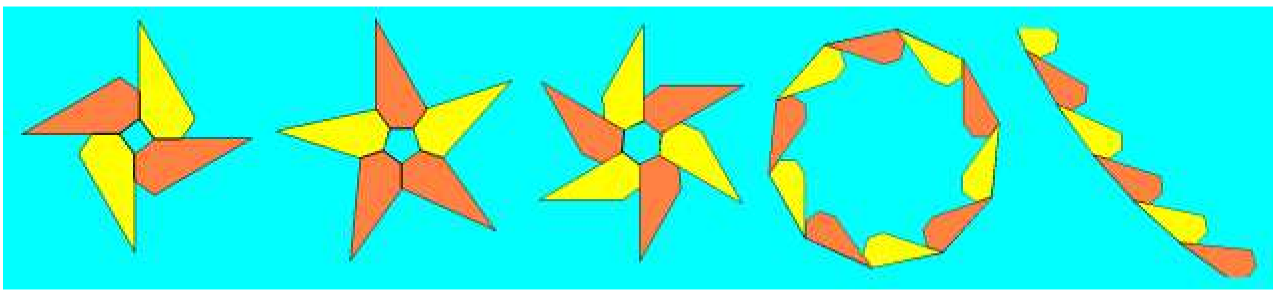

Figure 11

I showed this shape and its tilings to a colleague, whose comment was, "Does it tile on its own?" I swiftly convinced myself that it did not, even though there is some freedom in picking the lengths of the sides. "Maybe a hexagonal tile along these lines would tessellate?" I mused, as I coined some more terms. "Any n-sided Sider producing n different polygons in this way is a perfect Sider. If additionally this tile can tessellate on its own, it becomes a totally perfect Sider."

Thus so far we have a Sider (the quadrilateral), a perfect Sider (the pentagon), and a totally perfect Sider (the triangle). Is there a hexagonal totally perfect Sider? The equations became:

$$1 = (\pm\frac{1}{n_1} \ldots \pm\frac{1}{n_6}) \text{ and also}$$

$$a + b = 180 \pm \frac{360}{n_1},\ b + c = 180 \pm \frac{360}{n_2},\ c + d = 180 \pm \frac{360}{n_3},$$

$$d + e = 180 \pm \frac{360}{n_4},\ e + f = 180 \pm \frac{360}{n_5},\ f + a = 180 \pm \frac{360}{n_6}.$$

Adding the first, third, and fifth of these last six means that $\pm\frac{1}{n_1} \pm \frac{1}{n_3} \pm \frac{1}{n_5} = \frac{1}{2}$, while adding the second, fourth, and sixth means that $\pm\frac{1}{n_2} \pm \frac{1}{n_4} \pm \frac{1}{n_6} = \frac{1}{2}$.

How about $\frac{1}{3} + \frac{1}{8} + \frac{1}{24} = \frac{1}{2}$, and $\frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$? So the equations become:

$$a + b = 300,\ b + c = 270,\ c + d = 225,\ d + e = 240,\ e + f = 195, \text{ and } f + a = 210.$$

Solving these gives the six angles as a, 300 – a, a – 30, 255 – a, a – 15, and 210 – a.

Once again, an even number of sides seems to allow us a degree of freedom."So let's try a rough value for a, say 150°." This tile is indeed a perfect Sider, as expected. "But is it totally perfect?" I could almost get it to tile, but not quite...

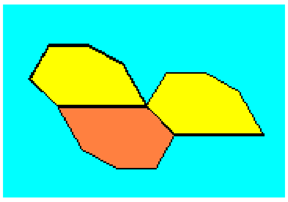

Figure 14

"Can I tweak it? If I insist that the three angles around this central point add to add to 360..."

This gave me a + (a – 15) + (210 – a) = 360 ⇒ a = 165°, the other angles being b = 135°, c = 135°, d = 90°, e = 150°, and f = 45°. Cutting these hexagons out takes a while, but it is worth it.

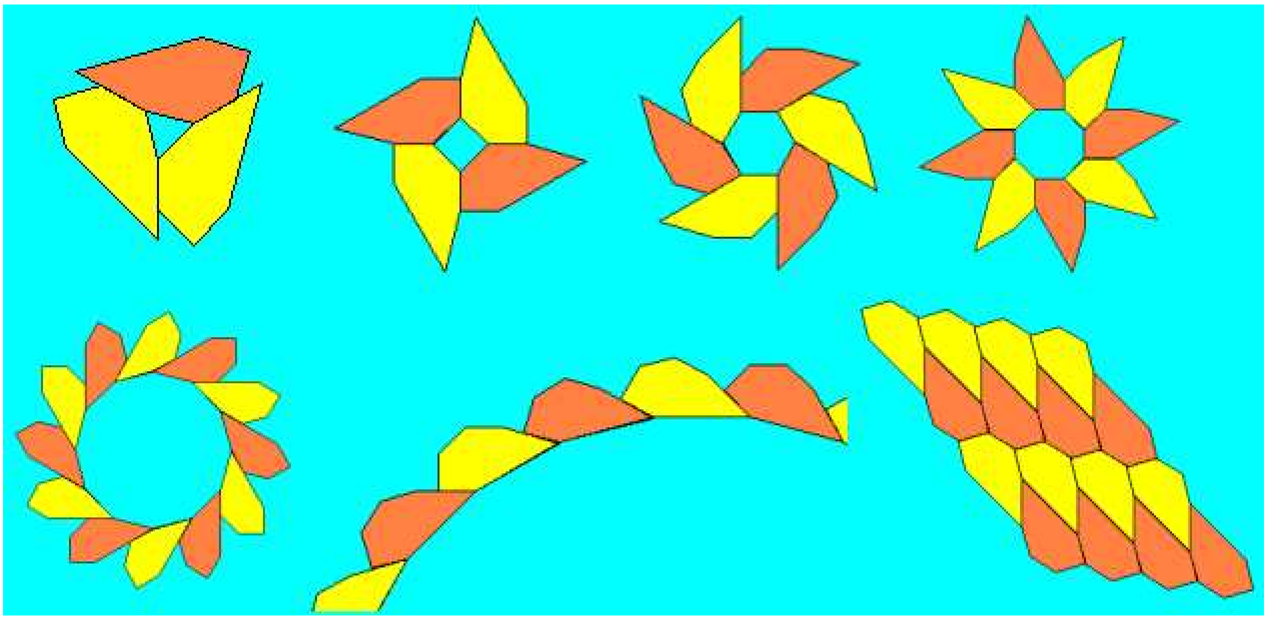

Figure 13

A totally perfect hexagonal Sider.

Using the fact that $\frac{3}{2} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{3.544...}$, it is possible to create a seven-sided Sider that generates the regular polygons from 3 through to 8 sides. (a = 165°, b = 135°, c = 135°, d = 117°, e = 123°, f = 108.4°, g = 116.6°) There is a pleasing opening-out effect as the number of sides grows.

Trying all this with students is a lot of fun. The pictures they can create when given a Sider to play with are beautiful, and if asked to calculate the angles rather than simply measure them, they are led gently into an implicit test: can you work out the interior or exterior angle of a regular polygon?

What are the most economical ways of drawing Siders in Logo? The practice with simple simultaneous equations in lots of variables is valuable too. Siders are surprising: an initially unremarkable hexagon has hidden depths. Maybe the same can be said of some of our students.

Keywords: tilings, tessellations, Logo, simultaneous equations, resistors in parallel, unit fractions

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