

Further Risp 1: The Twizzle

Level 1

Given a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the *twizzle* of $A = A^Z$ to be $\begin{pmatrix} b & d \\ a & c \end{pmatrix}$.

So the matrix A^Z is the matrix A with all its elements picked up and rotated anti-clockwise about the centre of the matrix through 90° .

Choose and write down a non-singular 2×2 matrix A , and then write down A^Z .

What's the determinant of A ? Of A^Z ? How are they connected? Does this always work?

Level 2

Suppose you define the twizzle of a 3×3 matrix A in a similar way.

Write down a non-singular 3×3 matrix A , together with A^Z .

Can you find $\det(A)$ and $\det(A^Z)$?

You could use a calculator to check your calculations.

How are $\det(A)$ and $\det(A^Z)$ connected here?

Can you prove any conjectures you make?

Level 3

Key trig facts; true for all θ in radians.

$$\begin{aligned}\sin \theta &= \cos \left(\frac{\pi}{2} - \theta \right), \cos \theta = \sin \left(\frac{\pi}{2} - \theta \right), \\ \sin(\pi - \theta) &= \sin \theta, \cos(\pi - \theta) = -\cos \theta, \\ \sin(-\theta) &= -\sin \theta, \cos(-\theta) = \cos \theta.\end{aligned}$$

Write down the 2×2 matrices for a rotation through θ about the origin, and for reflection in the line $y = x \tan \theta$.

Suppose you're given a 2×2 rotation matrix R ; can you say anything about what kind of matrix R^Z will be? How about R^{ZZ} ?

Suppose now you're given a 2×2 reflection matrix F ; can you say anything about what kind of matrix F^Z will be? How about F^{ZZ} ?

Level 4

If A^T stands for 'the transpose of A ', then the identity $(AB)^T = B^T A^T$ is true for all square matrices A and B of the same size (can you prove this?)

Can you show that $(AB)^{-1} = B^{-1}A^{-1}$ holds for all square matrices A and B of the same size?

In general, for all 2×2 matrices A and B , does $(AB)^Z = B^Z A^Z$?

If for all 2×2 matrices A you know that $(AB)^Z = B^Z A^Z$, what can you say about the 2×2 matrix B ? If for all 2×2 matrices B you know that $(AB)^Z = B^Z A^Z$, what can you say about the 2×2 matrix A ?

If for all 2×2 matrices A you know that $(AB)^Z = B^Z A^Z$ and $(BA)^Z = A^Z B^Z$, what can you say about the 2×2 matrix B ?