65. There’s always another prime...

Marcus de Sautoy said the other day, ‘If there is one piece of maths that I would like to teach all the world, it is that the number of prime numbers is infinite.’ I asked a Year 11 class recently how many primes there are, and someone said, slightly puzzled, ‘I mean, they’ve got to stop somewhere, haven’t they?’ Well, they certainly seem to get more and more spread out as our primes get bigger. We can have strings of whole numbers that are as long as we like that don’t feature a single prime – consider the numbers from n! + 2 to n! + n, none of which can be prime. But do they ever peter out completely?

Euclid gave us a proof of timeless simplicity two thousand years ago that the primes never peter out. Our students, however, may need a little preparation to fully appreciate its beauty. One fact is necessary – that all numbers can be broken down into primes. Once your students are happy with this, try the following: ‘Give me a whole number bigger than 1 that gives a remainder of 1 when we divide by 3 or 5 or 7.’

The smallest example is 106 – how can we find this without a lot of guessing? Well,

$106 = 3\times5\times7 + 1$, and once our students have understood the construction here, the sky is the limit. ‘Give me a whole number bigger than 1 that leaves a remainder of 1 when divided by 23 or 29 or 31?’ We can immediately say the smallest example is 20,678, as given by $23\times29\times31 + 1$.

Now we can take a further step. ‘I’m going to show you how, given a list of primes, we can always find another one’. Suppose our list to start with is 2, 3, 7, 11. Now consider the number $2\times3\times7\times11+1$. What is the remainder when we divide this number by 2? Clearly it is 1, and likewise for dividing by 3 and 7 and 11. In fact, the number is 463, which is prime, and so we’ve found a new prime for our list.

Will this always work? Let's try $2\times5\times7\times11 + 1$ – this is $771 = 3\times257$. So this time the number we make is not prime, but it is a product of primes, none of which are in our initial list. So once again, we have found a way to extend our list of primes.

Now we are ready to head down Euclid's road. Suppose the list of all primes is finite, so we can write them down: 2, 3, 5, 7,... p_n. Now let's make the number $2\times3\times5\times7... \times p_n + 1$. Certainly none of the primes in our list go into this – they all give remainder 1. So this number could be prime – in which case we have a new prime to add to our list. Or – it is the product of primes that are not in our list, so they are new too. So we have a contradiction (not allowed in maths!) and our initial thought that the list of primes is finite must be wrong. Delicious!

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