Associativity Sensitivity

Jonny Griffiths shares some thoughts around associativity and commutativity.

There has come a time in my mathematical career when I want to step back a little, become a bit more abstract in my thought, and consider things that either I have not really noticed before, or perhaps have taken for granted. For example, I observe that 2 + 5 = 5 + 2, and 2 × 5 = 5 × 2, and moreover, these swaps seem to work for any pair of real numbers. Does this phenomenon deserve a name? How about 'commutativity'? Then when I think about matrices, it turns out

A + B = B + A, but AB ≠ BA,

at least not in general. So apparently this idea of 'commutativity' is something I cannot take as an ever-present given.

Another phenomenon that has gradually impressed its importance on me is 'associativity'. Is the fact that 2 + (5 + 7) = (2 + 5) + 7 worthy of note? I observe that the sum 2 × (5 × 7) = (2 × 5) × 7 is also true and that I can generalise this, to the integers, the rationals, the reals and complex numbers. And with matrix multiplication, I remember that A(BC) = (AB)C, every time. Maybe 'associativity' always happens, whatever set of objects and operations I am working with? This turns out to be untrue; try

2 – (5 – 7) = (2 – 5) – 7.

But it's still an important idea. In fact, when the old mathematical masters came to defining a group, associativity was one of the four musts that they decided had to take place; no associativity, no group.

So, what happens when we comes to vectors? Vector addition is straightforwardly associative. What about vector multiplication? There are two standard ways to multiply two vectors in three dimensions;

- the dot product, $\underline{a}\,.\,\underline{b}$, that generates a scalar, and
- the cross product, $\underline{a} \times \underline{b}$, that generates a vector.

Thinking about associativity and the dot product. Does $\underline{a}\,.\,(\underline{b}\,.\,\underline{c}) = (\underline{a}\,.\,\underline{b})\,.\,\underline{c}$? This question makes no sense to me; the left hand side and the right hand side both try to dot together a scalar and a vector, which is not allowed. The question concerning what happens with the vector product, however, is more nuanced.

I remember that $\underline{a} \times \underline{b}$ can be defined as $|\underline{a}||\underline{b}|\sin\theta\hat{\underline{n}}$, where θ is the angle between $\underline{a}$ and $\underline{b}$, and where $\hat{\underline{n}}$ is the unit vector perpendicular to both $\underline{a}$ and $\underline{b}$ (using a right-hand screw action from $\underline{a}$ to $\underline{b}$ to give the direction). It transpires that if

$$\underline{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \text{ and } \underline{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \text{ then } \underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

If $\underline{a}$ and $\underline{b}$ are parallel, then $\theta = 0$ and $\underline{a} \times \underline{b}$ is the zero vector. We have from the definition that $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$; we say that × is *anti-commutative*. Another useful fact; the cross product distributes over addition, which means that for all $\underline{a}$, $\underline{b}$ and $\underline{c}$,

$$\underline{a}\times(\underline{b}+\underline{c}) = \underline{a}\times\underline{b}+\underline{a}\times\underline{c}.$$

So, to our problem; when is it true that $\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\times\underline{c}$? If you try this for yourself (and please feel free to have a go now), you might agree with me that it's a deceptively tricky question...

Let us try to head towards a solution. I am going to try to make the main part as geometrical as possible, just lines and planes, although later there will be other methods.

We are going to be thinking in three dimensions - the non-zero vector $\underline{a}\times\underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$, after all. It's clear that $\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\times\underline{c}$ holds if $\underline{a}$, $\underline{b}$ or $\underline{c}$ is the zero vector, so we will exclude these possibilities.

I can see that $\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\times\underline{c}$ is not generally true. Picking three vectors $\underline{a}$, $\underline{b}$ or $\underline{c}$ more or less at random gives

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\times\left(\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}5\\-10\\5\end{pmatrix}, \quad \left(\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix}\right)\times\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}4\\-7\\7\end{pmatrix}.$$

No equality here. But it is sometimes true. Here is an example;

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\times\left(\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\2\\3\end{pmatrix}\right)=\begin{pmatrix}9\\-24\\13\end{pmatrix}, \quad \left(\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix}\right)\times\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}9\\-24\\13\end{pmatrix}$$

Suppose $\underline{a}$ is parallel to $\underline{c}$, so $\underline{a} = \lambda\underline{c}$. Then

$$\begin{aligned}\underline{a}\times(\underline{b}\times\underline{c}) &= \lambda\underline{c}\times(\underline{b}\times\underline{c})\\ &= -\lambda(\underline{b}\times\underline{c})\times\underline{c}\\ &= \lambda(\underline{c}\times\underline{b})\times\underline{c}\\ &= (\lambda\underline{c}\times\underline{b})\times\underline{c}\\ &= (\underline{a}\times\underline{b})\times\underline{c}.\end{aligned}$$

In this case, it is always true that $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \times \underline{c}$, whatever $\underline{b}$ is. I can get *Derive* (a computer algebra package) to do a check;

$$\#1: \begin{bmatrix} p\cdot\lambda \\ q\cdot\lambda \\ r\cdot\lambda \end{bmatrix} \times \left(\begin{bmatrix} s \\ t \\ u \end{bmatrix} \times \begin{bmatrix} p \\ q \\ r \end{bmatrix}\right) - \begin{bmatrix} p\cdot\lambda \\ q\cdot\lambda \\ r\cdot\lambda \end{bmatrix} \times \begin{bmatrix} s \\ t \\ u \end{bmatrix} \times \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

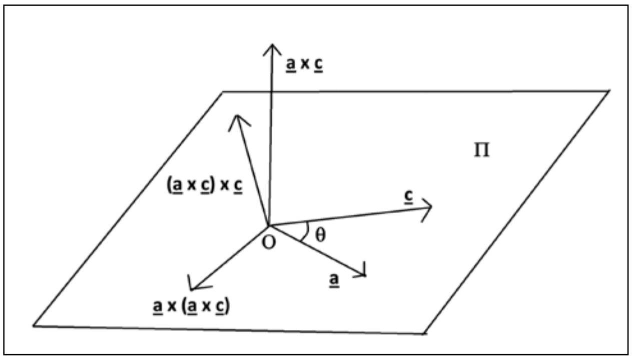
$$\#2: \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Derive does not include the brackets in the case $(\underline{a} \times \underline{b}) \times \underline{c}$).

The other possibility is that $\underline{a}$ is not parallel to $\underline{c}$. Thus $\underline{a}$ and $\underline{c}$ define a plane Π through O, and $\underline{a} \times \underline{c}$ is perpendicular to Π. I can take $\underline{a}$, $\underline{c}$ and $\underline{a} \times \underline{c}$ as a basis for vectors in $\mathbb{R}^3$, and write

$$\underline{b} = \lambda\underline{a} + \mu\underline{c} + \delta\underline{a} \times \underline{c}\ .$$

Now since $\underline{a} \times \underline{c}$ is perpendicular to both $\underline{a}$ and $\underline{c}$, we can see that $\underline{a} \times (\underline{a} \times \underline{c})$ and $(\underline{a} \times \underline{c}) \times \underline{c}$ both lie in Π.



Now I have

$$\begin{aligned}\underline{a}\times(\underline{b}\times\underline{c}) &= \underline{a}\times[(\lambda\underline{a}+\mu\underline{c}+\delta(\underline{a}\times\underline{c}))\times\underline{c}]\\ &= \underline{a}\times[\lambda\underline{a}\times\underline{c}+\mu\underline{c}\times\underline{c}+\delta(\underline{a}\times\underline{c})\times\underline{c}]\\ &= \lambda\underline{a}\times(\underline{a}\times\underline{c})+\delta(\underline{a}\times((\underline{a}\times\underline{c})\times\underline{c})),\end{aligned}$$

which means $\underline{a} \times (\underline{b} \times \underline{c})$ is the sum of λ times (a vector in Π) and δ times (a vector in the direction of $(\underline{a} \times \underline{c})$.

Similarly, I have $(\underline{a} \times \underline{b}) \times \underline{c} = \mu(\underline{a} \times \underline{c}) \times \underline{c} + \delta(\ \underline{a} \times (\underline{a} \times \underline{c})\ ) \times \underline{c}$, and so $(\underline{a} \times \underline{b}) \times \underline{c}$ is the sum of μ times (a vector in Π) and δ times (a vector in the direction of $\underline{a} \times \underline{c}$).

But if $\underline{v} = \underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \times \underline{c}$ then $\underline{v}$ must be perpendicular to both $\underline{a}$ and $\underline{c}$, that is, $\underline{v}$ must be a vector in the direction of $\underline{a} \times \underline{c}$.

That must mean $\lambda = \mu = 0$, and so $\underline{b} = \delta\ \underline{a} \times \underline{c}$.

There is one thing left for me to check; that $\underline{a} \times (\ (\underline{a} \times \underline{c}) \times \underline{c})$ and $(\underline{a} \times (\underline{a} \times \underline{c})\ ) \times \underline{c})$ are the same. Looking at the diagram, they both have the same direction, which is that of $\underline{a} \times \underline{c}$. I use the diagram to find the magnitudes of both expressions. They both turn out to be $a^2 c^2 \sin\theta\cos\theta$, and so I am done; if $\underline{a}$ and $\underline{c}$ are not parallel, then $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \times \underline{c}$ holds if and only if $\underline{b} = \delta\ \underline{a} \times \underline{c}$.

Now to the short cuts. There are a couple of general vector identities that speed the above deduction up, but which perhaps provide less geometrical insight along the way.

Firstly, there is the *Jacobi identity*; for any three vectors a, b and c,

$$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = \underline{0}.$$

This tells me that

$$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) - (\underline{a} \times \underline{b}) \times \underline{c} = \underline{0}, \text{ and so}$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) = \underline{c} \Leftrightarrow \underline{b}\times (\underline{c} \times \underline{a}) = \underline{0}.$$

Thus either $\underline{a}$ and $\underline{c}$ are parallel, or $\underline{b}$ is parallel to $\underline{a} \times \underline{c}$.

Secondly, there is *the identity for the vector triple product* (which is what I have on each side of $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{a} \times (\underline{b} \times \underline{c})$). For any three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{b})\ \underline{b} - (\underline{a}.\underline{b})\ \underline{c}$.

So

$$\begin{aligned}\underline{a}\times(\underline{b}\times\underline{c}) - (\underline{a}\times\underline{b})\times\underline{c} = \underline{0} &\Leftrightarrow \underline{a}\times(\underline{b}\times\underline{c}) + \underline{c}\times(\underline{a}\times\underline{b}) = \underline{0}\\ &\Leftrightarrow (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c} + (\underline{c}.\underline{b})\underline{a} - (\underline{c}.\underline{a})\underline{b} = \underline{0}\\ &\Leftrightarrow (\underline{c}.\underline{b})\underline{a} = (\underline{a}.\underline{b})\underline{c}.\end{aligned}$$

This holds if $\underline{a} = \lambda\ \underline{c}$, or if $\underline{a}.\underline{b}$ and $\underline{b}.\underline{c}$ are both zero (so $\underline{b}$ is in the direction of $\underline{a} \times \underline{c}$).

Thankfully the multiple approaches converge onto the same solution. To sum up; there are essentially two situations where vector cross product associativity holds. Either

- $\underline{a}$ and $\underline{c}$ are parallel, or
- $\underline{b}$ is perpendicular to the plane defined by $\underline{a}$ and $\underline{c}$.

So ,either

$$\underline{a} = \begin{pmatrix} \lambda c_1 \\ \lambda c_2 \\ \lambda c_3 \end{pmatrix}, \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ or } \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \underline{b} = \lambda \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \lambda \begin{pmatrix} a_2c_3 - a_3c_2 \\ a_3c_1 - a_1c_3 \\ a_1c_2 - a_2c_1 \end{pmatrix}.$$

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