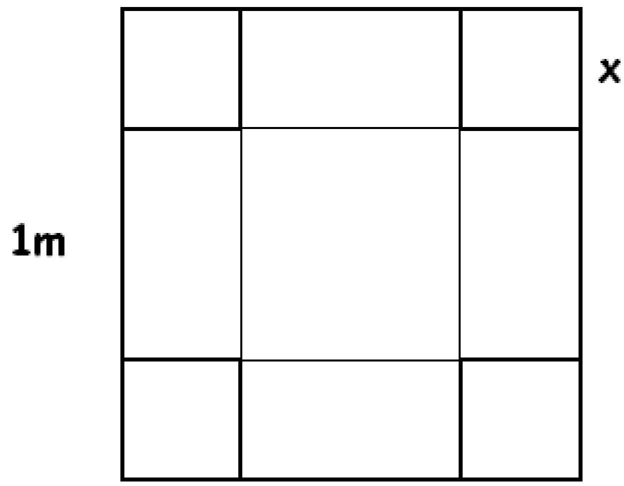
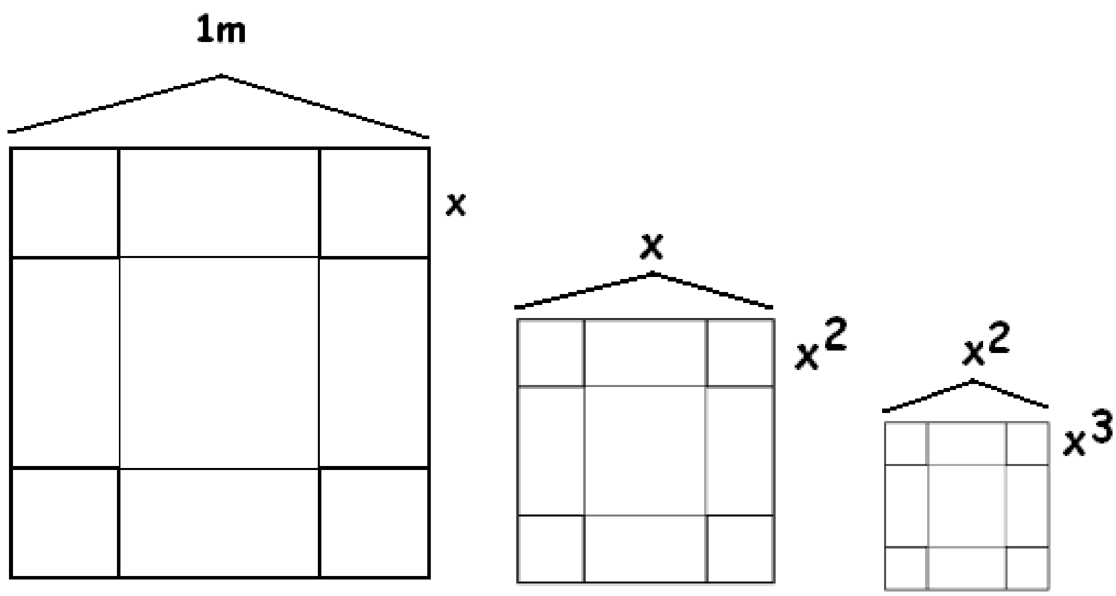
The Box Problem Extended



The Box Problem is a classic. Given a square sheet of metal, side 1m, what is the maximum possible volume of the lid-less box you can now fold by cutting four equal squares from the corners of the sheet? This provides an excellent answer to the question, “But what use is differentiation?” The answer turns out to be about 0.0741 m^3, given when x = 1/6m.

It occurred to me recently that no use is made of the offcuts in this presentation, which seems a waste. Suppose the box-maker gathered up the square corners each time, and cuts these, in the same proportions as she cut the initial square, to make more similar boxes, and does the same to the offcuts from these, and so on. Which value of x gives the maximum total volume for all the boxes now?



Before tackling the mathematics, would you guess that the answer is:

a) less than 1/6 b) equal to 1/6 c) greater than 1/6?

It is not hard to see from the diagram that

$$\mathbf{V_{total} = (1\text{-}2x)^2x + 4(x\text{-}2x^2)^2\, x^2 + 16\, (x^2 - 2\, x^3)^2\, x^3 + ...}$$

$$\mathbf{= (1\text{-}2x)^2x + 4(1\text{-}2x)^2x^4 + 16\, (1\text{-}2x)^2x^7 + ...}$$

This is a geometric series, with common ratio $\mathbf{4x^3}$,
so to infinity its sum is **a/(1-r),** that is, $\mathbf{(1\text{-}2x)^2x/(1\text{-}4x^3)}$

Now, differentiating using the quotient rule,

$$\frac{dV_{total}}{dx} = \frac{(1-4x^3)(1-8x+12x^2)-12x^2(1-2x)^2x}{(1-4x^3)^2}$$

Putting this equal to 0 leaves us trying to solve

$$\mathbf{16x^4 - 8x^3 - 12x^2 + 8x - 1 = 0}$$

Homing in on the root with the help of a calculator gives x = 0.1737, slightly more than 0.16667, or a sixth.

So the maximum value of $\mathbf{V_{total}}$ in this new situation is $\mathbf{0.07556\ m^3}$:
taking x as 1/6 gives $\mathbf{V_{total} = 0.07547\ m^3}$, a tiny difference.

This approach to the box problem gives no waste metal at all,
and a lot of Russian-doll-type boxes!

One question that suggests itself as a result of this: suppose we remove the restriction that the offcuts have to be cut in the same proportions as the original. Can we improve on the figure of $\mathbf{0.07556\ m^3}$?

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