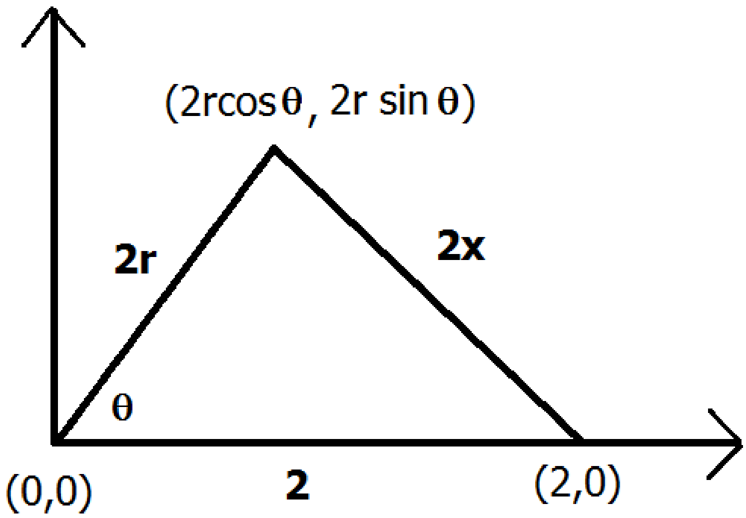
# The Centre of Mass Problem

*A uniform triangular lamina ABC has its centre of mass at G.*

*The lamina is now removed, and A, B and C are joined by uniform rods to create a triangle.*

*The centre of mass is now at H.*

*If the point G coincides with the point H, show that ABC is equilateral or a straight line.*




Let the triangle be as above, where r > 0 and 0 < θ < π, without loss of generality. By the Cos Rule, $x = \sqrt{r^2 + 1 - 2r\cos\theta}$.

For the lamina, the centre of mass is at

$$\frac{1}{3}\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2r\cos\theta \\ 2r\sin\theta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} \dfrac{2+2r\cos\theta}{3} \\ \dfrac{2r\sin\theta}{3} \end{pmatrix}.$$

For the three rods, the centre of mass is at:

$$\frac{1}{2+2r+2x}\left[2\begin{pmatrix}1\\0\end{pmatrix}+2r\begin{pmatrix}r\cos\theta\\r\sin\theta\end{pmatrix}+2x\begin{pmatrix}r\cos\theta+1\\r\sin\theta\end{pmatrix}\right]$$

$$=\begin{pmatrix}\dfrac{r^2\cos\theta+1+x(r\cos\theta+1)}{1+r+x}\\ \dfrac{r^2\sin\theta+rx\sin\theta}{1+r+x}\end{pmatrix}.$$

So for the x-coordinates of the centres of mass to coincide,

$$\frac{2+2r\cos\theta}{3}=\frac{r^2\cos\theta+1+x(r\cos\theta+1)}{1+r+x},$$

which gives

$$2r\cos\theta+2r-r^2\cos\theta-1=x(r\cos\theta+1).$$

Squaring, substituting for x and simplifying gives

$$2r^3\cos^3\theta-4r^3\cos^2\theta-6r^3\cos\theta+7r^2\cos^2\theta$$
$$+10r^2\cos\theta+4r\cos\theta+3r^2-4r=0$$

which factorises to

$$r(1+\cos\theta)(2r\cos\theta-1)(r\cos\theta-3r+4)=0.$$

We know that $r > 0$, and $0 < \theta < \pi$, so therefore

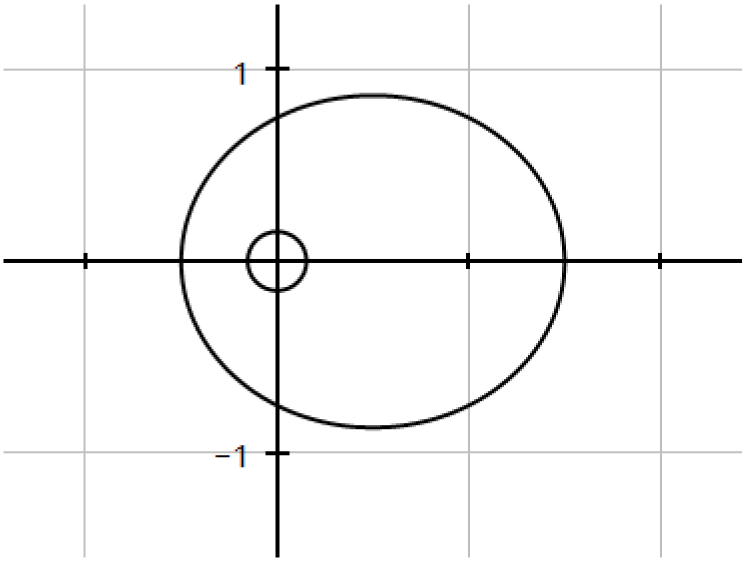
$$r=\frac{1}{2\cos\theta} \quad \text{or} \quad r=\frac{4}{3-\cos\theta}.$$

For the y-coordinates of the centres of mass to be equal,

$$\frac{2r\sin\theta}{3} = \frac{r^2\sin\theta + xr\sin\theta}{1+r+x}.$$

This simplifies to 2 – r = x, which gives $r = \dfrac{3}{4-2\cos\theta}$.

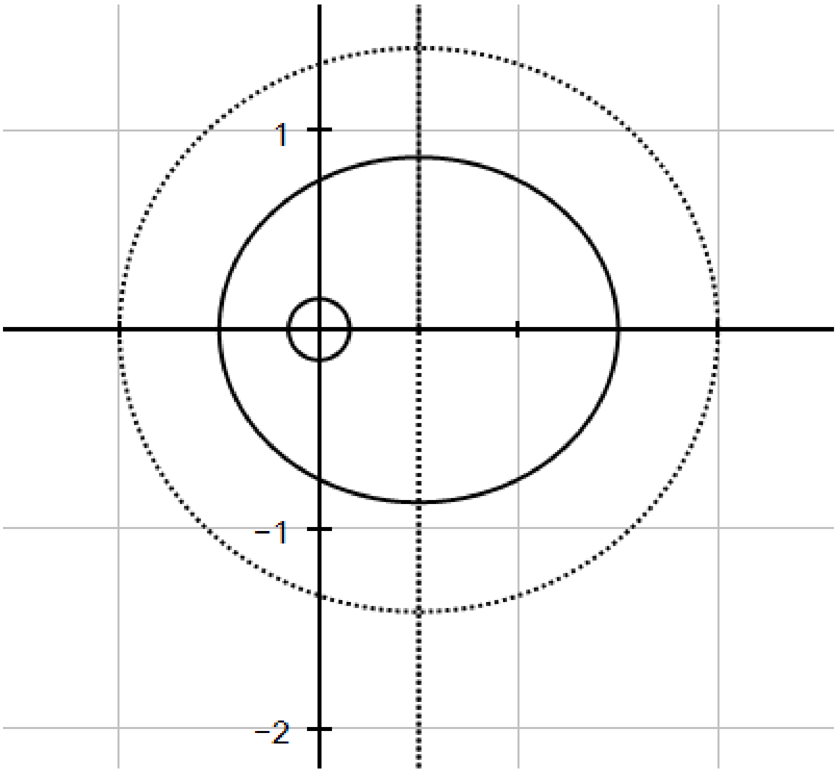
The graph of this is an ellipse:




Any viable solution must be an intersection point of this and one of

$$r = \frac{1}{2\cos\theta} \quad \text{or} \quad r = \frac{4}{3-\cos\theta}.$$

These graphs are added below in dotted lines.




Thus our only solution is where $r = \dfrac{1}{2\cos\theta}$ and $r = \dfrac{3}{4 - 2\cos\theta}$,

which gives cos θ = 0.5, and r = 1.

Thus the equilateral triangle and the straight line are the only solutions for ABC.

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