Identities Connecting The Chebyshev Polynomials, $T_n(x), U_n(x), V_n(x)$, and $W_n(x)$.

Author: Jonny Griffiths, February 2016.

Introduction

There are many families of polynomials in mathematics, and they often occur naturally in pairs. The Fibonacci polynomials and the Lucas polynomials, for example, are generated by the same recurrence relation but with different starting values, and there are many identities that link the two families [1]. The same is true for the Chebyshev polynomials of the first and second kinds, $T_n(x)$ and $U_n(x)$ [2], respectively. There are two further polynomial families that are less well-known, the Chebyshev polynomials of the third and fourth kinds, $V_n(x)$ and $W_n(x)$ [3], respectively. Each of the four kinds is an example of an orthogonal polynomial family $P_n(x)$, where for some appropriate weight function $W(x)$, $\int_{-1}^{1} P_n(x)P_m(x)W(x)dx = 0$ whenever $n \neq m$. The families $T_n(x)$ and $U_n(x)$ in particular are ubiquitous in their mathematical uses, in approximation theory, in differential equations, and in solving the Pell equation, to name but three. There are also many connections between $T_n(x), U_n(x)$, $V_n(x)$ and $W_n(x)$, some of which are explored here, and some of which we hope are new.

Chebyshev polynomials of the first, second, third and fourth kinds

The polynomial family $T_n(x)$ can be found by writing $\cos(n\theta)$ as a polynomial in $\cos\theta$, and then replacing each $\cos\theta$ by x (in what follows, θ is always $\arccos(x)$). We thus have $T_n(\cos\theta) = \cos(n\theta)$, or $T_n(x) = \cos(n(\arccos(x)))$. The early polynomials $T_n(x)$ are given in Table 1 (the triangle given by the coefficients is A028297 at [4]). These polynomials may also be defined by the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad \text{with} \quad T_0(x) = 1, T_1(x) = x.$$

The Chebyshev polynomials of the second kind ($U_n(x)$) are generated by

$T_0(x)$	1	$U_0(x)$	1
$T_1(x)$	x	$U_1(x)$	$2x$
$T_2(x)$	$2x^2-1$	$U_2(x)$	$4x^2-1$
$T_3(x)$	$4x^3-3x$	$U_3(x)$	$8x^3-4x$
$T_4(x)$	$8x^4-8x^2+1$	$U_4(x)$	$16x^4-12x^2+1$
$T_5(x)$	$16x^5-20x^3+5x$	$U_5(x)$	$32x^5-32x^3+6x$
$T_6(x)$	$32x^6-48x^4+18x^2-1$	$U_6(x)$	$64x^6-80x^4+24x^2-1$
$T_7(x)$	$64x^7-112x^5+56x^3-7x$	$U_7(x)$	$128x^7-192x^5+80x^3-8x$

Table 1: Chebyshev polynomials of the first ($T_n(x)$) & second ($U_n(x)$) kinds

the same recurrence, but with different starting values. We have

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad \text{with} \quad U_0(x) = 1, U_1(x) = 2x.$$

The initial $U_n(x)$ are given in Table 1 (the triangle given by the coefficients is A133156 at [4]). There is a trigonometric link here too, since

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

The Dirichlet kernel, $D_n(\theta)$, is used in Fourier analysis, and is defined by

$$D_n(\theta) = \frac{\sin\frac{(2n+1)\theta}{2}}{\sin\frac{\theta}{2}} = 1 + 2\sum_{k=1}^{n}\cos(k\theta).$$

This tells us that we can can express $D_n(\theta)$ as a polynomial in $\cos\theta$. In a similar way, $\dfrac{\cos\frac{(2n+1)\theta}{2}}{\cos\frac{\theta}{2}}$ can also be written as a polynomial in $\cos\theta$ (this is easily shown by induction). We define

$$V_n(x) = \frac{\cos\frac{(2n+1)\theta}{2}}{\cos\frac{\theta}{2}} \quad \text{and} \quad W_n(x) = \frac{\sin\frac{(2n+1)\theta}{2}}{\sin\frac{\theta}{2}},$$

where again θ is $\arccos(x)$. These families are known as the Chebyshev polynomials of the third and fourth kinds, respectively, and are once again generated by the same recurrence as $T_n(x)$ and $U_n(x)$, but with different starting values (this is easily shown by induction). We have

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x), \quad \text{with} \quad V_0(x) = 1, V_1(x) = 2x - 1,$$

$$W_{n+1}(x) = 2xW_n(x) - W_{n-1}(x), \quad \text{with} \quad W_0(x) = 1, W_1(x) = 2x + 1.$$

The polynomial family $V_n(x)$ starts as in Table 2 (the triangle given by the coefficients is A228565 at [4]), while the polynomials $W_n(x)$ begin as in Table 3 (the triangle given by the coefficients is A180870 at [4]). Exploration

$V_0(x)$	1
$V_1(x)$	$2x - 1$
$V_2(x)$	$4x^2 - 2x - 1$
$V_3(x)$	$8x^3 - 4x^2 - 4x + 1$
$V_4(x)$	$16x^4 - 8x^3 - 12x^2 + 4x + 1$
$V_5(x)$	$32x^5 - 16x^4 - 32x^3 + 12x^2 + 6x - 1$
$V_6(x)$	$64x^6 - 32x^5 - 80x^4 + 32x^3 + 24x^2 - 6x - 1$
$V_7(x)$	$128x^7 - 64x^6 - 192x^5 + 80x^4 + 80x^3 - 24x^2 - 8x + 1$

Table 2: Chebyshev polynomials of the third kind, $V_n(x)$

$W_0(x)$	1
$W_1(x)$	$2x + 1$
$W_2(x)$	$4x^2 + 2x - 1$
$W_3(x)$	$8x^3 + 4x^2 - 4x - 1$
$W_4(x)$	$16x^4 + 8x^3 - 12x^2 - 4x + 1$
$W_5(x)$	$32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1$
$W_6(x)$	$64x^6 + 32x^5 - 80x^4 - 32x^3 + 24x^2 + 6x - 1$
$W_7(x)$	$128x^7 + 64x^6 - 192x^5 - 80x^4 + 80x^3 + 24x^2 - 8x - 1$

Table 3: Chebyshev polynomials of the fourth kind, $W_n(x)$

reveals several simple identities connecting these families, for example,

$$U_n(x) + U_{n-1}(x) = \frac{\sin((n+1)\theta)}{\sin\theta} + \frac{\sin(n\theta)}{\sin\theta} = \frac{2\sin\frac{(2n+1)\theta}{2}\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = W_n(x),$$

while similarly

$$U_n(x) - U_{n-1}(x) = V_n(x), \quad \text{and}$$

$$T_n(x) = \frac{V_n(x) + V_{n-1}(x)}{2} = \frac{W_n(x) - W_{n-1}(x)}{2}, \quad \text{and}$$

$$U_n(x) = \frac{V_n(x) + W_n(x)}{2}.$$

We also have

$$U_n(x) = \sum_0^n V_r(x), W_n(x) = 2\sum_0^n T_r(x) - 1.$$

Additionally

$$V_n(x) = \frac{\cos\frac{(2n+1)\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\sin(\frac{\pi}{2} - \frac{(2n+1)\theta}{2})}{\sin(\frac{\pi}{2} - \frac{\theta}{2})}$$

$$= \frac{\sin(n\pi + \frac{(2n+1)(\pi-\theta)}{2})}{\sin(\frac{\pi-\theta}{2})} = (-1)^n W_n(-x),$$

since if $\arccos(y) = \pi - \arccos(x)$, then $y = -x$. Immediately from this, we have

$$W_n(x) = (-1)^n V_n(-x).$$

A notation for recurrence relations

A recurrence relation may be represented in more than one way. Let $x, y, f(x, y), \ldots$ represent the sequence of numbers generated by the order-2 recurrence relation

$$u_{n+1} = f(u_{n-1}, u_n) \quad \text{for} \quad n \geqslant 1, \quad \text{where} \quad u_0 = x, u_1 = y.$$

For simplicity, we will say that that $x, y, f(x, y), \ldots$ is synonymous with this recurrence and the sequence it generates. The infinite sequence $u_0, u_1, u_2, \ldots$ will be written as (u_n). We call $f(x, y)$ the *recurrence function*.

One particular recurrence

Consider the sequence

$$\coth\theta, \coth\phi, \coth(\theta - \phi), \coth(2\phi - \theta), \coth(2\theta - 3\phi), \ldots.$$

or

$$\coth u_0, \coth u_1, \coth u_2, \coth u_3, \coth u_4, \ldots.$$

where $u_{n+1} = u_{n-1} - u_n$ for all $n \geqslant 1$, with $u_0 = \theta, u_1 = \phi$. Given that

$$\coth(\theta - \phi) = \frac{\coth\phi\coth\theta - 1}{\coth\phi - \coth\theta},$$

this sequence is the recurrence $x, y, \dfrac{xy-1}{y-x}, \ldots$ with $x = \coth\theta, y = \coth\phi$. We will now place the function coth to one side, and concentrate on this recurrence to which it gives rise.

If (u_n) is the recurrence $x, y, \dfrac{xy-1}{y-x}, \ldots,$ with $x = u_0, y = u_1$, then we define an *H-sequence* to be $[u_r, u_{r+1}, \ldots, u_s]$, where $0 \leqslant r < s$. If $u_r, u_{r+1}, \ldots, u_s$ are all integers, we call this an *integer H-sequence*. (The 'H' stands for 'Hikorski' – see [5]). It is easily checked that $[2, 3, 5, 7, 17]$ is an integer H-sequence. The question arises, what is the longest possible integer H-sequence? Some work on the computer shows that integer H-sequences are almost exclusively of length 2, 3 and 4, and in being of length 5, the H-sequence $[2, 3, 5, 7, 17]$ is the anomaly rather than the rule.

Conjecture: With the exception of $[2, 3, 5, 7, 17]$, no integer H-sequence contains more than four elements.

This remains open at the time of writing. It is easy to verify that

$$[n, 2n - 1, 2n + 1, 2n^2 - 1]$$

will always be an H-sequence – let us call H-sequences of this type *basic H-sequences*. Can the term before or the term after a basic integer H-sequence ever be an integer here, thus giving a quintet akin to $[2, 3, 5, 7, 17]$? It is easy to prove it cannot for $n > 2$ ([5], page 187). Not all H-sequences, however, are basic – Table 4 gives a list of the smallest non-basic H-sequences with four elements such that terms are positive and increasing.

Searching for the integers in the first and fourth rows within the *Online Encyclopedia of Integer Sequences* [4] is revealing. There is clearly an intimate connection (see Table 5) with the number triangle associated with $T_n(x)$, the Chebyshev polynomials of the first kind (the sequence at OEIS is A101124). Can we clarify this link?

u_0	7	17	26	31	49	71	97	97	99	127	161	199
u_1	11	29	41	55	89	131	153	181	169	239	305	379
u_2	19	41	71	71	109	155	265	209	239	271	341	419
u_3	26	99	97	244	485	846	362	1351	577	2024	2889	3970

Table 4: Early H-sequences, increasing with all terms positive

$x \rightarrow$ $n \downarrow$	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	$\cdots$
2	-1	1	7	17	31	49	71	97	$\cdots$	$\cdots$
3	0	1	26	99	244	485	846	$\cdots$	$\cdots$	$\cdots$
4	1	1	97	577	1921	4801	$\cdots$	$\cdots$	$\cdots$	$\cdots$
5	0	1	362	3363	15124	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
6	-1	1	1351	19601	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
7	0	1	5042	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
8	1	1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
9	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 5: The start of the triangle of values given by $T_n(x)$, with $n, x, \geqslant 0, n + x \leqslant 9$

Connecting the Chebyshev polynomials

It is easily checked that if $[j, a, b, k]$ is an H-sequence, then

$$[-j, -a, -b, -k], \quad \left[\frac{1}{j}, a, \frac{1}{b}, \frac{1}{k}\right], \quad \left[j, \frac{1}{a}, \frac{1}{b}, k\right], \quad \left[\frac{1}{j}, \frac{1}{a}, b, \frac{1}{k}\right]$$

are too. We can also check straightforwardly that if $\dfrac{pq+1}{p+q}$ is the first term of an H-sequence, and if p is the second, then the next two terms are q and

$\dfrac{pq-1}{q-p}$, giving us the H-sequence

$$\left[\frac{pq+1}{p+q}, p, q, \frac{pq-1}{q-p}\right].$$

What happens now if we put $j = \dfrac{pq+1}{p+q}, k = \dfrac{pq-1}{q-p}$ here? This yields

$$p = \frac{jk+1+\sqrt{(1-j^2)(1-k^2)}}{j+k}, q = \frac{jk-1+\sqrt{(1-j^2)(1-k^2)}}{k-j}, \qquad (1)$$

or

$$p = \frac{jk+1-\sqrt{(1-j^2)(1-k^2)}}{j+k}, q = \frac{jk-1-\sqrt{(1-j^2)(1-k^2)}}{k-j}. \qquad (2)$$

and so the quartets

$$\left[j, \frac{jk+1+\sqrt{(1-j^2)(1-k^2)}}{j+k}, \frac{jk-1+\sqrt{(1-j^2)(1-k^2)}}{k-j}, k\right]$$

and

$$\left[j, \frac{jk+1-\sqrt{(1-j^2)(1-k^2)}}{j+k}, \frac{jk-1-\sqrt{(1-j^2)(1-k^2)}}{k-j}, k\right]$$

are H-sequences. (These quartets are of the form $[j, a, b, k]$ and $\left[j, \frac{1}{a}, \frac{1}{b}, k\right]$).
The expression $\sqrt{(1-j^2)(1-k^2)}$ strongly suggests a trigonometrical substitution. If we put

$$j = \cos(n\theta), k = \cos((n+1)\theta) \qquad (3)$$

into (2), then

$$p = \frac{jk+1-\sqrt{(1-j^2)(1-k^2)}}{j+k} = \frac{\cos(n\theta)\cos((n+1)\theta) - \sin(n\theta)\sin((n+1)\theta)+1}{\cos(n\theta)+\cos((n+1)\theta)}$$

$$= \frac{\cos((2n+1)\theta)+1}{2\cos\frac{\theta}{2}\cos\frac{(2n+1)\theta}{2}} = \frac{2\cos^2\frac{(2n+1)\theta}{2}}{2\cos\frac{\theta}{2}\cos\frac{(2n+1)\theta}{2}} = \frac{\cos\frac{(2n+1)\theta}{2}}{\cos\frac{\theta}{2}} = V_n(x)$$

Similarly

$$q = \frac{jk - 1 - \sqrt{(1-j^2)(1-k^2)}}{k - j} = \frac{\cos(n\theta)\cos((n+1)\theta) - \sin(n\theta)\sin((n+1)\theta) - 1}{\cos((n+1)\theta) - \cos(n\theta)}$$

$$= \frac{\cos((2n+1)\theta) - 1}{-2\sin\frac{\theta}{2}\sin\frac{(2n+1)\theta}{2}} = \frac{-2\sin^2\frac{(2n+1)\theta}{2}}{-2\sin\frac{\theta}{2}\sin\frac{(2n+1)\theta}{2}} = \frac{\sin\frac{(2n+1)\theta}{2}}{\sin\frac{\theta}{2}} = W_n(x).$$

We have now that $[T_n(x), V_n(x), W_n(x), T_{n+1}(x)]$ is an H-sequence for all n and x, and if x is an integer, then it will be an integer H-sequence. We thus have

$$W_n(x) = \frac{T_n(x)V_n(x) - 1}{V_n(x) - T_n(x)}, \quad \text{and} \quad T_{n+1}(x) = \frac{V_n(x)W_n(x) - 1}{W_n(x) - V_n(x)}, \quad \text{for} \quad n \geqslant 0.$$

We hope these pleasingly symmetric identities are new. Substituting (3) into (1) gives that $\left[T_n(x), \frac{1}{V_n(x)}, \frac{1}{W_n(x)}, T_{n+1}(x)\right]$ is also an H-sequence for all n and x.

Related infinite sequences

Now consider the infinite sequence

$$T_1(x), V_1(x), W_1(x), T_2(x), V_2(x), W_2(x), T_3(x), V_3(x), \ldots.$$

Let us call this sequence $S(x)$. If $f(x,y) = \frac{xy-1}{y-x}$, then we know from our work above that

$$f(T_n(x), V_n(x)) = W_n(x), \quad \text{and} \quad f(V_n(x), W_n(x)) = T_{n+1}(x),$$

but, as it transpires,

$$f(W_n(x), T_{n+1}(x)) \neq V_{n+1}(x).$$

Can we find a recurrence function $g(x,y)$ so that $g(W_n(x), T_{n+1}(x)) = V_{n+1}(x)$? If we assume that $g(x,y)$ is of the form $\frac{axy + bx^2 + cy^2 + dx + ey + f}{pxy + qx^2 + ry^2 + sx + ty + u}$, then experimenting with the help of a computer algebra package swiftly shows that $g(x,y) = \dfrac{xy+2y^2-1}{x+y}$ (a proof is given in [5], page 195).

So suppose we take the recurrences $x, y, f(x,y) = \dfrac{xy-1}{y-x}, \ldots$ and $x, y, g(x,y) = \dfrac{xy-2y^2+1}{y-x}, \ldots$. Starting with x and y as our first two terms, we can iterate 'f twice then g once'. If we note that $x = T_1(x)$, and substitute for y the expression $2x-1 = V_1(x)$, we then have the sequence $S(x)$, namely

$$x, 2x-1, 2x+1, 2x^2-1, 4x^2-2x-1, 4x^2+2x-1, 4x^3-3x, 8x^3-4x^2-4x+1, \ldots,$$

Suppose we do not substitute $2x-1$ for y; the sequence (let's call it (u_n)) is now

$$x, y, \frac{xy-1}{y-x}, \frac{x(y^2+1)-2y}{2xy-y^2-1}, \frac{2x+y(y^2-3)}{2xy-y^2-1},$$

$$\frac{2x^2-xy(y^2+3)+3y^2-1}{(x-y)(2xy-y^2-1)}, \frac{4x^3-8x^2y+x(y^4+6y^2-3)-4y(y^2-1)}{(2xy-y^2-1)^2}, \ldots.$$

If we consider (u_{3k+2}), putting $p = u_2, q = u_5, f(p,q) = u_8$, then it can be shown that $p, q, f(p,q), \ldots$ forms a recurrence relation (we could call this the Composition Technique - see [5], page 30). In this case, eliminating x and y gives $f(p,q) = \dfrac{p+q^3-2q}{pq-1}, \ldots$, which yields

$$V_{n+1}(x) = \frac{V_n(x)^3 - 2V_n(x) + V_{n-1}(x)}{V_n(x)V_{n-1}(x) - 1}, V_0(x) = 1, V_1(x) = 2x-1. \quad (4)$$

The same technique on (u_{3k}) yields

$$W_{n+1}(x) = \frac{W_n(x)^3 - 2W_n(x) - W_{n-1}(x)}{W_n(x)W_{n-1}(x) + 1}, W_0(x) = 1, W_1(x) = 2x+1. \quad (5)$$

A generalisation

At (3) we chose $j = \cos(n\theta), k = \cos((n+1)\theta)$. There is a generalisation we can make here; choosing $j = \cos(ab\theta), k = \cos((a+1)b\theta)$ yields the H-sequence

$$\left[\cos(ab\theta), \frac{\cos\frac{(2a+1)b\theta}{2}}{\cos\frac{b\theta}{2}}, \frac{\sin\frac{(2a+1)b\theta}{2}}{\sin\frac{b\theta}{2}}, \cos((a+1)b\theta)\right].$$

We consider two cases here, as b is even or odd.

Case 1: $b = 2c$

Here we have the H-sequence

$$\left[\cos(2ac\theta), \frac{\cos((2a+1)c\theta)}{\cos(c\theta)}, \frac{\sin((2a+1)c\theta)}{\sin(c\theta)}, \cos(2(a+1)c\theta)\right],$$

which is

$$\left[T_{2ac}(x), \frac{T_{(2a+1)c}(x)}{T_c(x)}, \frac{U_{(2a+1)c-1}(x)}{U_{c-1}(x)}, T_{2(a+1)c}(x)\right].$$

Rayes, Trevisan, and Wang [6] have shown that $T_n(x)|T_{kn}(x)$ and $U_{c-1}(x)|U_{kc-1}(x)$ for all $n, k, c \in \mathbb{N}$, so these are all integer polynomials. In particular for $c = 1$ we have the H-sequence

$$\left[T_{2a}(x), \frac{T_{2a+1}(x)}{x}, U_{2a}(x), T_{2(a+1)}(x)\right].$$

We can now consider the infinite sequence

$$(v_n) = T_0(x), \frac{T_1(x)}{x}, U_0(x), T_2(x), \frac{T_3(x)}{x}, U_2(x), T_4(x), \ldots.$$

Applying the Composition Technique to the sequence $(v_{3k+1}), k \geqslant 0$, gives

$$p, q, \frac{q^3 - 2q + p}{pq - 1}, \ldots.$$

We thus have the connection between the polynomials $T_k(x)$ for k odd,

$$T_{2n+1}(x) = \frac{T_{2n-1}(x)^3 - 2x^2 T_{2n-1}(x) + x^2 T_{2n-3}(x)}{T_{2n-1}T_{2n-3} - x^2}.$$

The same technique applied to (v_{3k+2}) gives a connection between the polynomials $U_k(x)$ for k even,

$$U_{2n+2}(x) = \frac{U_{2n}(x)^3 - 2U_{2n}(x) - U_{2n-2}(x)}{U_{2n}U_{2n-2} + 1}.$$

The similarity of these two recurrences to (4) and (5) is notable.

Case 2: $b = 2c + 1$

Here we have the H-sequence

$$\left[\cos((2c+1)a\theta), \frac{\cos\frac{(2a+1)(2c+1)\theta}{2}}{\cos\frac{(2c+1)\theta}{2}}, \frac{\sin\frac{(2a+1)(2c+1)\theta}{2}}{\sin\frac{(2c+1)\theta}{2}}, \cos((a+1)(2c+1)\theta)\right].$$

Dividing the second term here top and bottom by $\cos\frac{\theta}{2}$, and dividing the third term top and bottom by $\sin\frac{\theta}{2}$, this becomes

$$\left[T_{(2c+1)a}(x), \frac{V_{2ac+a+c}(x)}{V_c(x)}, \frac{W_{2ac+a+c}(x)}{W_c(x)}, T_{(a+1)(2c+1)}(x)\right].$$

This suggests that

$$V_c(x)|V_{2ac+a+c}(x), \quad W_c(x)|W_{2ac+a+c}(x), a, c \in \mathbb{N}.$$

Experimenting with small values of a and c supports this. It is hoped that the divisibility properties of the Chebyshev polynomials of the third and fourth kinds might form the basis of future research.

Conclusion

The polynomial families $T_n(x)$ and $U_n(x)$ are well-known, and their interrelationships have been thoroughly explored. The polynomial families $V_n(x)$ and $W_n(x)$ are, it would seem, less well-known; the OEIS [4] does not appear to contain any related sequences except for those contributed by the author. It's hoped the identities given here will encourage further exploration, and that in the future the polynomial families $V_n(x)$ and $W_n(x)$ might accumulate a wider significance.

Bibliography

[1] Wikipedia. Fibonacci Polynomials. **http://en.wikipedia.org/wiki/Fibonacci_polynomials**. (accessed February 2016).

[2] Wikipedia. Chebyshev Polynomials. **http://en.wikipedia.org/wiki/Chebyshev_polynomials**. (accessed February 2016).

[3] J. C. Mason. Factorisation properties of Chebychev polynomials. *Mathematics*, Volume 49, Issues 1 to 3, December 1993, Pages 169 to 178.

[4] Online Encyclopedia of Integer Sequences. **http://oeis.org**. (accessed February 2016).

[5] J. Griffiths. *Lyness Cycles, Hikorski Triples and Elliptic Curves.* MSc Thesis, University of East Anglia, 2012, **http://www.s253053503.websitehome.co.uk/jg-msc-uea/index.html**, (accessed February 2016).

[6] M. Rayes, V. Trevisan, and P. Wang. Factorisation properties of Chebychev polynomials. *Comput. Math. Appl.*, **50** (2005) 1231–1240.

Article author:

Jonny Griffiths, **hello@jonny-griffiths.net**, February 2016
20 Rosebery Road, Norwich, NR3 3NA,
01603 423090