The Circle of Squares

Jonny Griffiths and Tom Grogan share a mathematical task which fascinated them.

It is always nice when a new mathematical task that is remarkably addictive arrives in one's inbox. I received an example from my friend Len Fisher the other day;

> **For which *n* can you put the numbers 1 to *n* in some order into a circle so that every adjacent pair of numbers adds to a square?**

I would urge you to try this now if you have not met it before. You could come back to what is below later and see if your thoughts coincide with mine.

My first thought was to construct a gentler version of the game - let's leave squares to one side for the moment. A simple variation is to ask;

> **For which *n* can you put the numbers 1 to *n* in some order into a circle so that every adjacent pair of numbers adds to an even number?**

A little thought suggests this is always impossible. Odd numbers have to be next to odd numbers, and even numbers have to be next to even numbers, but if you try to put them into a circle, there will have to be an odd number next to an even number somewhere, which means no circle.

What happens if we change 'even' to 'odd'? This looks better – if n is even, then we can alternate odd and even numbers around our circle (how many possible circles are there in this case?) But if n is odd, then we are back to no circle, since that extra odd number has nowhere to go.

This could be a good way into the problem with younger learners, kicking off valuable ideas about proof.

Now back to the original question. How to start? Small values for n don't seem to supply possible solutions. Rules may start to form in our heads…

Rule 1: Think about the number n in the circle, with neighbours a and b. If the circle is to work, then

a + n = a square and b + n = a square.

This means there must be at least two squares in the interval [n+1, 2n-1].

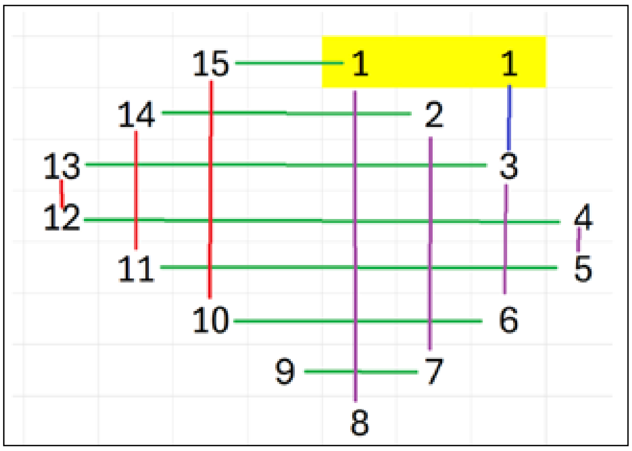
This rules out n = 1 to 12 and n = 16 to 18 straight away.

Rule 2: This is similar; the two squares that some number a < n can produce with its two neighbours must lie in the interval [a + 1, a + n]. So, putting a = 9, n = 13 to 15 are ruled out; we only have one square that fits the conditions in each case.

Rule 3: There is a problem with a = 18, since 18 cannot link with 18 to get 36, so we must have at least three squares in the interval [19, 18 + n]. This rules out n = 19 to 30.

There is a similar rule 3 for a = 8; we must have at least three squares in the interval [9, 8 + n], so n is at least 17. Whenever we have an even square, rule 3 in some shape or form will apply.

So, we have established that the smallest possible number n that can create our required circle is 31. It is worth noticing that 15 is a near miss. We can arrange our numbers helpfully here like this;



We can see this as a graph, with the numbers as vertices, and two vertices linked with an edge if they add to a square. So, we have one blue edge corresponding to a sum of 4, four purple edges that give 9, seven green edges that give 16, and three red edges that give 25. The vertices 3 and 1 have degree 3, 9 and 8 have degree 1, while all the others are degree 2. In looking for a circle as above, we are seeking a Hamiltonian cycle, that is, a closed trail along the edges that visits every vertex once, but which does not necessarily use every edge. We can see that this is impossible for n = 15; if our trail visits 8 or 9, as it must, there is no way out. But we do have a connected graph that is semi-Eulerian (exactly two vertices have odd degree); we can create a trail containing all the vertices that starts at 8 and finishes at 9.

8-1-15-10-6-3-13-12-4-5-11-14-2-7-9

So, if only 8 linked to 9, we would have our circle.

Let us turn to the number 31, the smallest number for

	1	4	9	16	25	36	49				
1		3	8	15	24						
2		2	7	14	23						
3		1	6	13	22						
4			5	12	21						
5			4	11	20	31					
6			3	10	19	30					
7			2	9	18	29			18	7	29
8			1	8	17	28					
9				7	16	27			16	9	27
10				6	15	26					
11				5	14	25					
12				4	13	24					
13				3	12	23					
14				2	11	22					
15				1	10	21					
16					9	20			9	16	20
17					8	19			8	17	18
18					7	18	31		7	18	31
19					6	17	30				
20					5	16	29		16	20	29
21					4	15	28				
22					3	14	27				
23					2	13	26				
24					1	12	25				
25						11	24		11	25	24
26						10	23		10	26	23
27						9	22		9	27	22
28						8	21		8	28	21
29						7	20		7	29	20
30						6	19		6	30	19
31						5	18		5	31	18

n that we have not ruled out. Excel is a big help; what follows is an incomplete version of the spreadsheet that can be assembled. See Table 1

If there is a possible circle, parts of it will be forced (the chains of three on the right) where we know the two neighbours that a number has to have. The orange numbers 'have been dealt with', the green ones are half-committed (two greens makes an orange), while the blue ones are impossible. The number 9 has 7 greyed out, since 7 already appears twice in green in the chains on the right. We can start to join the chains together to make larger ones. It takes a while to get a feel for how this all fits together, but I found it remarkable how Excel was much easier to work with than pen and paper.

My conclusion? I could find no circle, but I could find a trail containing every number.

6-30-19-17-8-28-21-4-12-13-3-1-24-25-11-5-31-18-7-29-20-16-9-27-22-14-2-23-26-10-15

If you've found a circle, I'd be delighted to hear about it!

It's noticeable that if we ask Excel to help us with this square problem for larger n, the circle seems easier to find. If we look at n = 48, for example, there is very little that is forced (32 is the only such number) so creating Hamiltonian cycles becomes relatively easy (there are lots of them). This suggests that there will be some kind of 'sweet spot' between 31 and 48 where we have a unique possible circle. It turns out that the number 32 does the job – making n just a tiny bit bigger than 31 adds that little extra freedom we need. In fact, you can introduce the initial problem like this if you wish;

> **Put the numbers 1 to 32 in some order into a circle so that every adjacent pair of numbers adds to a square.**

Checking that this circle is unique is not easy – I found myself generating vast tree diagrams along the way.

So, what can we vary in this problem? We have seen that we can vary n, for sure, but as we saw with the 'odd' and 'even' versions of the problem, we can also vary the type of number that the circle aims for.

How many squares are there between n and 2n - 1? If 2n - 1 is roughly x^2, then n is roughly $x^2/2$. Suppose we are given a natural number m. When does $(x - m)^2$ lie between n and 2n - 1? As x gets large, it is the x^2 term that dominates when we multiply out $(x - m)^2$. This must exceed n (which is roughly $x^2/2$) as x increases beyond a certain value. So m can be as large as we please, if x can be arbitrarily large, potentially giving us as many squares to link with as we wish. This argument works just as well with cubes or fourth powers and so on.

This leads to…

> **Conjecture; Looking at cubes, or fourth powers, or other powers; if n is large enough, we will always be able to make the required circle.**

We could hopefully make the argument above rigorous with a bit of work.

It is tempting to change our type of number to Fibonacc'. Is a Fibonacci circle possible? Certainly, the number n = 20 gives us a near miss; we can create a trail containing every number that works.

17-4-9-12-1-20-14-7-6-15-19-2-11-10-3-18-16-5-8-13

Every adjacent pair adds to a Fibonacci number. But Fibonacci numbers increase more quickly than a power of n, since if F_m is the m^th^ Fibonacci number m^{th}, then

$$F_m = \frac{\phi^m - (1-\phi)^m}{\sqrt{5}}$$ where $\phi = \frac{1+\sqrt{5}}{2}$ (the Golden Ratio).

The value m here is up as a power. This table helps us to see the different growth rates;

n	squares up to n	cubes up to n	Fibonacci up to n
100	10	4	11
1000	31	10	16
10000	100	21	20
100000	316	46	25
1000000	1000	100	30

> **Conjecture; a Fibonacci circle is impossible for all n.**

I know that $\frac{F_{m+1}}{F_m}$ tends to ϕ as m tends to infinity

So, if n is roughly F_m, then the next two Fibonacci numbers are roughly ϕF_m and $\phi^2 F_m$, where $\phi^2>2$; in other words, the second Fibonacci number is out of reach.

Imagine n = 100. Then 55 will (pleasingly) link with 34 and 89, which are both Fibonacci numbers. But then 89 links only with 55 and 0 (not allowed), and this situation can't be rectified by increasing or decreasing n.

Now to the most important bit of all; what happened when I tried this with my students? One of my students, Tom offered this analysis of his thought processes.

Tom

At first, I wanted to see what numbers "worked well together" and if there were any pairs or triplets that could be confirmed as together for the sequence to work. I found that there are eight separate triplets that have to be together (for example, [4,32.17]).

This example works because these are the only two numbers that 32 goes with to make a square number. If you continue this pattern and look at 31 you find that it only goes with 5 and 18.

This goes on for eight pairs, the last one being [11 which goes with 25 and 24]. We have now accounted for 24 out of 32 of the numbers.

For further progress, you could consider numbers that fit in between 2twoof these pairs. For example, 16 fits nicely between 9 and 20 (which are both part of separate triplets). We now have a string of [22-27-9-16-20-29-7].

You can then consider what numbers could pair with the ends; 22 goes with 3 and 14, 7 goes with 9 and 2 and so on. Do this with all the strings you have so far, and use a process of elimination to connect the strings until you have the final sequence;

[1,8,28,21,4,32,17,19,30,6,3,13,12,24,25,11,5,31,18,7,29,20,16,9,27,22,14,2,23,26,10,15]

The trial-and-error aspect of the last part of this method was slightly unsatisfying, and I am sure there is a more mathematical solution, but the problem was interesting, and overcoming this challenge was an exciting feeling. After coming to my final solution I was interested to see if you could do it with other numbers, for example 1 to 100. I found a forum on ***math.stackexchange.com*** -

https://math.stackexchange.com/questions/4320839/proving-the-number-circle-internet-meme-where-sum-of-each-adjacent-numbers-is-a

that suggested it was possible to make a number circle like this for all numbers from 32 to 100; the question is, does this pattern stop, or could you make such circles for all numbers larger 32? This seems like an interesting challenge that could be worth researching. Who knows, this might become one of the great unsolved problems?

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Concluding thoughts

Tom and his compadres added (confessed?) that there had been a 'race' element to solving the problem that was part of the enjoyment. All in all I had the satisfying feeling of kicking off a craze. Such tasks, that can swiftly generate excellent mathematical motivation with very little work from the teacher, are immensely valuable.

With thanks to Harry Eyers, Noah Head and all the other students who worked on this problem.