When the Coefficients are the Roots

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Abstract: when are the coefficients of a polynomial equation equal to its roots?

The other day I ran into this question: *if the roots of the quadratic equation* $x^2 + ax + b = 0$ *are* a *and* b*, what are the possible values for* a *and* b*?*

The question is straightforwardly solved. We say $x^2+ax+b = (x-a)(x-b) = x^2 - (a+b)x + ab$. This is true for all values of x, so we can equate the coefficients of each side, giving $-a-b = a$, and $ab = b$. Thus on solving, the possibilities for (a, b) are $(0, 0)$ or $(1, -2)$.

So far, so good, but the question to my mind immediately invited generalisation. If we're given the equation $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0$, then what possibilities for $(a_{n-1}, a_{n-2}, \cdots, a_0)$ are there if these values are also the roots of the equation?

We can note firstly that $(0, 0, \cdots, 0)$ will always be a solution (we'll call this the trivial solution).

Secondly, if $(a_{n-1}, a_{n-2}, \cdots, a_0)$ is a solution for the degree-n equation, then by multiplying our degree-n equation by x, we see $(a_{n-1}, a_{n-2}, \cdots, a_0, 0)$ will be a solution in the degree-$(n+1)$ case. Thus if we add zeroes as we go, the number of solutions we find will be non-decreasing as n increases.

What happens with the degree-3 case, the cubic equation? We have $x^3 + ax^2 + bx + c = (x-a)(x-b)(x-c)$, and on expanding and comparing coefficients, we find

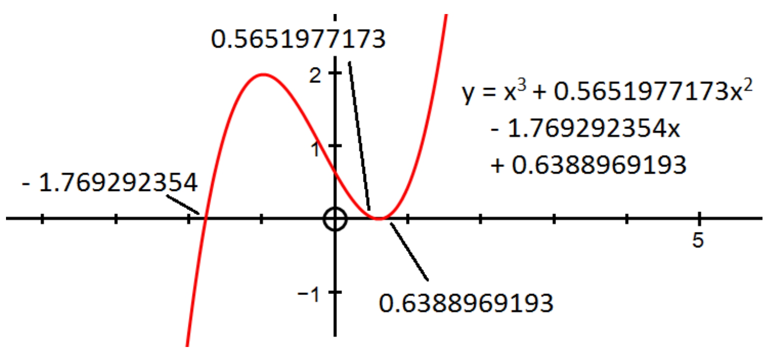
$$[2a + b + c, ab + ac + bc - b, c(ab + 1)] = [0, 0, 0].$$

Let me explain the notation here. The algebra rapidly becomes unfriendly with this problem, and the use of a computer algebra package becomes essential. I'm using Derive 6, but there are many alternatives – if you don't currently use such a program, it will open up a wealth of maths investigations to you if you do. When it comes to algebra, Derive is infinitely quicker and more accurate than I can ever hope to be.

Here $[x, y, z]$ denotes a three-dimensional vector, a format which makes the substitution of values within Derive easy. We effectively have three equations in three unknowns. Solving $2a + b + c = 0$ gives $c = -2a - b$, and substituting in, we have $[0, -2a^2 - 2ab - b^2 - b, -(2a + b)(ab + 1)] = [0, 0, 0]$. Now putting $(2a + b)(ab + 1) = 0$, we have $b = -2a$ or $b = -\dfrac{1}{a}$.

Taking the first of these options gives $[0, 2a(1 - a), 0] = [0, 0, 0]$, and so $a = 0$ or 1. These give us the trivial solution $(0, 0, 0)$, and $(1, -2, 0)$. Taking $b = -\dfrac{1}{a}$ instead gives $\left[0, -2a^2 + \dfrac{1}{a} - \dfrac{1}{a^2} + 2, 0\right]$, and solving for a here gives four solutions, two complex (we'll ignore these), $a = 1$ and (curiously) $a = 0.5651977173....$ Substituting back, we get the triplets for (a, b, c) of $(1, -1, -1)$ and $(0.5651977173..., -1.769292354..., 0.6388969193...)$.

So we have four real solutions to our degree-3 problem. It's pleasing to ask a graphing program to plot this last possibility.



Can we make a conjecture now? Doing this on the basis of $n = 2$ and 3 alone seems rash. But it does seem appropriate to ask, will there always be one solution to the degree-n equation where all the coefficients are non-zero? (We have two such solutions here.) And what happens to these coefficients as n gets larger - will they perhaps form a sequence that tends to a limit?

On to the quartic. We have (with the added zeroes) four solutions already - is there a new one to be found, where none of the coefficients are zero? We have

$$x^4 + ax^3 + bx^2 + cx + d = (x - a)(x - b)(x - c)(x - d).$$

Derive tells us from this that

$$[2a + b + c + d, (a(b + c + d) + b(c + d - 1) + cd),$$

$$(a(b(c + d) + cd) + c(bd + 1)), d(abc - 1)] = [0, 0, 0, 0].$$

We have $d = -2a - b - c$, and so

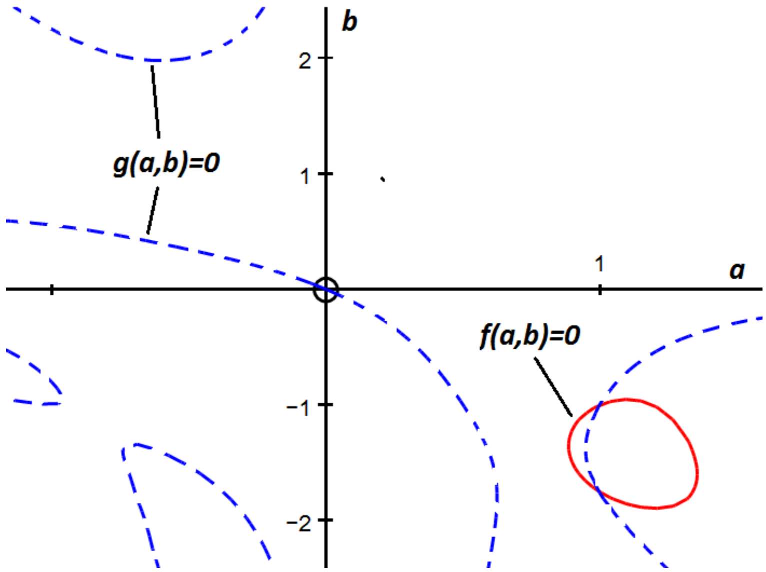
$$[0, -2a^2 - a(2b+2c) - b^2 - b(c+1) - c^2, -2a^2(b+c) - a(b^2+3bc+c^2) - c(b^2+bc-1),$$

$$(2a + b + c)(1 - abc)] = [0, 0, 0, 0].$$

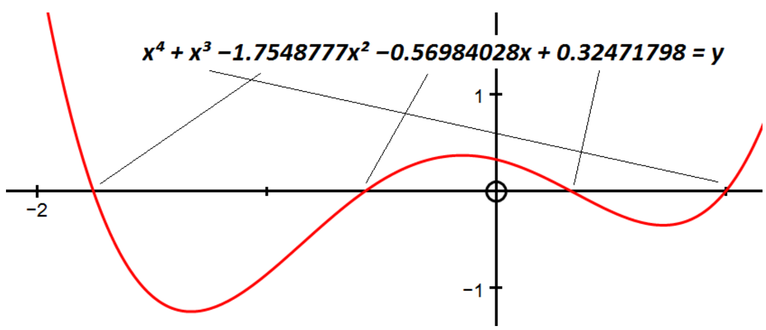
From the final element, we see that $c = -2a - b$, or $c = \frac{1}{ab}$. Substituting in $c = -2a - b$ leads to solutions we have already with $d = 0$ added, that's (0,0,0,0), (1,-2,0,0), (1,-1,-1,0), and (0.5651977173... , -1.769292354..., 0.6388969193..., 0). But $c = \frac{1}{ab}$ is more interesting. This yields

$$\left[0, -\frac{f(a, b)}{a^2b^2}, -\frac{g(a, b)}{a^2b^2}, 0\right]$$

where f is a degree-6 polynomial, and g is degree-7. Derive is certainly earning its keep, but asking it to solve $f(a, b) = 0$ or $g(a, b) = 0$ for a or b, however, sees it down tools. We will need a different strategy. How about plotting the curves $f(a, b) = 0$ and $g(a, b) = 0$? Their intersection points will then give us what we need. This leads to (using Autograph) a closed curve for $f(a, b) = 0$, and a more widely-spread curve for $g(a, b) = 0$.



The uppermost of the intersection points is $(1, -1)$, which leads to the solution $(1, -1, -1, 0)$, which we have already. But the lower point at $(1, -1.7548777)$ gives us a fresh solution, that is, $(1, -1.7548777..., -0.56984028..., 0.32471798...)$.



So our conjecture has proved true for $n = 4$ at least, although this time we have only a single solution for $(a_{n-1}, a_{n-2}, \cdots, a_0)$ that is completely non-zero. Now on to the quintic, which initially solves much as before. We have $e = -2a - b - c - d$, and on substituting, we have $d = -2a - b - c$, which leads on to solutions we already have, or $d = -\dfrac{1}{abc}$, which gives us three polynomials in a, b and c that must equate to zero. The hope is that we can now move into three dimensions, and plot three (fiendish-looking) surfaces; the points where they intersect should give us more possible values for a, b and c. The power of Autograph is evident here – it plots the first two surfaces without a murmur. But try to plot the third, and – nothing. It's as if we're asking for a plot for $(x^2 + y^2 + z^2 + 1)(x^2y^2z^2 + 1) = 0$ – there are simply no real points that work.

And so my conjecture fails, and not for the first time, the quintic proves to be a stumbling block. We are, in the degree-5 case it seems, asking for too much. Our new revised conjecture must be that for degrees higher than four, we can never find a solution for $(a_{n-1}, a_{n-2}, \cdots, a_0)$ where all the a_i are non-zero. Maybe someone out there is au fait with a package even more powerful than Derive - I would be delighted to hear of a counter-example if so.

Jonny Griffiths taught mathematics at Paston Sixth Form College in Norfolk for over twenty years. He has studied mathematics and education at Cambridge University, the Open University, and the University of East Anglia. Possible claims to fame include being a member of Harvey and the Wallbangers, *a popular band in the Eighties, and playing the character Stringfellow on the childrens' television programme* Playdays. *He currently works for Underground Mathematics.*