Cycling in Task Design

How would you tackle this question?

Working in three dimensions, you are given the three points

A = (a, b, c), B = (b, c, a) and C = (c, a, b), where a, b and c are distinct real numbers.

The area of the triangle ABC is 1; what's its perimeter?

While trying to hatch a plan for this problem, we might feel a bit glum. We have three unknowns (a, b and c), and just one equation connecting them (the area fact). Surely we will need more information? We can but shrug, hope something nice happens, and get started. And something nice does happen! We find

$$AB = \sqrt{(a-b)^2 + (b-c)^2 + (c-a)^2} = BC = CA\ .$$

So the triangle ABC is equilateral, and there is in effect only one unknown, its side-length d. Now

$$\frac{1}{2}d^2 \sin 60^o = 1 \Rightarrow d^2 = \frac{4}{\sqrt{3}} \Rightarrow d = \frac{2}{\sqrt{\sqrt{3}}}.$$

Thus the perimeter of the triangle ABC is $\dfrac{6}{\sqrt{\sqrt{3}}}$.

How does the niceness arise here? From the way the values a, b and c cycle around in the question. From my experience of writing maths tasks, this trope reliably produces questions with a neat twist.

Colin Foster's etude idea, that it's possible to compose exercises to develop key mathematical skills that are beautiful in their own right, sits happily alongside this cycling technique. Suppose you want three questions to set your students as exploratory practice; cycling three parameters may well give you these, with the process additionally offering a connection that your students can work towards, a mini-theorem if you like.

Pick three numbers a, b and c (all different, with none equal to 0 or 1).

Now write down and solve the following equations using your numbers:

x + a = bx + c

x + b = cx + a

x + c = ax + b.

Charlotte says that if she is given two of the solutions

(without being told a, b and c) she can predict the third.

She says her rule is $\text{Solution 3} = -\dfrac{\text{Solution 1} + \text{Solution 2}}{(\text{Solution 1} \times \text{Solution 2}) + 1}$ **.**

Does this work for your numbers? For those of your colleagues?

If so, can you prove it will always work?

Answer at http://s253053503.websitehome.co.uk/cycling/equations.pdf

And if you want six questions, then looking at the six possible orders for three distinct values may give you something helpful (for example, see Risp 3 and Risp 17 at www.risps.co.uk).

There is sometimes a down side to this type of question; the algebra can get off-putting. Certainly this three-equation problem above eventually requires algebra that could be taxing when done by hand. So instead, you could introduce students to the power of a computer algebra package to do the heavy lifting for you. My preferred system is Derive, as sold by Texas Instruments (see www.chartwellyorke.com/ti-nspire). There is also a basic CAS system built into Geogebra, which is freely available, and my guess is this will grow rapidly stronger as time goes by. Using CAS, we're now free to ask a whole range of questions that we could not sensibly tackle before. And you may have to field the question, 'If a computer can do all this for us in the blink of an eye, why do we have to battle so hard with it on our own?'

How would you define 'a theorem'? The phrase that plays around in my mind currently is 'an explained coincidence'. It seems that cycling values as above throws up coincidences naturally, and in such a way as to pique student curiosity. That curiosity can lead on to a need for proof, and on to proof itself using generalisation. And all the while vital skills are being practiced en passant.

Of course, your cycle could include more than three values. How about this question, also in three dimensions, that cycles four values;

What's the condition for the four points

A = (a, b, c), B = (b, c, d), C = (c, d, a) and D = (d, a, b),

where a, b, c and d are distinct non-zero real values,

to be coplanar?

Students could pick integer values for a, b and c and then try to find d. However they do this, the resulting value of d is (remarkably) then constrained to be an integer too. Solving a cubic is required, however, which is not impossible but which is tough; wheeling in CAS is probably a good idea. This starts by being unpromisingly complicated, but the final result is a treat.

Answer at http://s253053503.websitehome.co.uk/cycling/coplanar.pdf

Any task that includes numbers can be included. How about matrices? Take this question (to be set after some hard matrix study and edited according to what your students have covered).

Pick three positive integer values for p, q and r, and write down the three matrices

$$A=\begin{pmatrix} p & q \\ 0 & r \end{pmatrix}, B=\begin{pmatrix} r & p \\ 0 & q \end{pmatrix}, C=\begin{pmatrix} q & r \\ 0 & p \end{pmatrix}.$$

What transformation does the matrix A + B + C represent?

What transformation does the matrix ABC represent?

What kind of number is |ABC|? Can you explain this?

What transformation does the matrix AB – C^2 represent?

Can you generalise these results?

Suppose we drop the condition that we have to be working with positive integers here.

Does the set of matrices $\left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in \mathbf{R} \right\}$ **with matrix multiplication form a group?**

If not, how could we tweak this set so that it does?

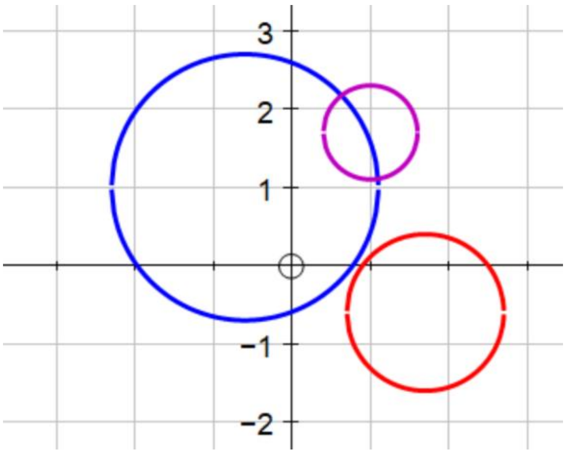
Answer at http://s253053503.websitehome.co.uk/cycling/matrices.pdf

I'm not suggesting we bombard our students with cycling questions every lesson. Matching activities, for example, certainly have their place, and they have become very popular, but they can be overdone. 'Death by Matching' is a danger in many A Level classrooms. But cycling seems to throw up a different kind of answer each time; predicting solutions to cycling questions is not always easy.

My favourite cycling problem of all time is this. Suppose we plot these three circles in a graphing package.

$$(x-a)^2 + (y-b)^2 = c^2$$
$$(x-b)^2 + (y-c)^2 = a^2$$
$$(x-c)^2 + (y-a)^2 = b^2$$

We now vary a, b and c and watch the circles ebb and flow. Something very noticeable and rather eerie takes place; we can never get the circles to be completely disjoint. In other words, there is always at least one point that is shared between at least two of the circles.



This becomes

The Friendly Circle Theorem

If A, B and C are the three circles above, then $(A \cap B) \cup (B \cap C) \cup (C \cap A) \neq \varnothing$ **.**

Can you prove it?

Answer at http://s253053503.websitehome.co.uk/cycling/friendly.pdf

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