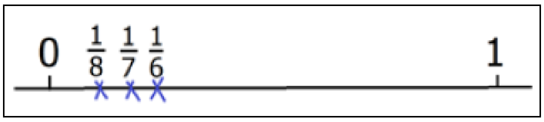
The fraction question

A conversation between Jonny Griffiths and a subset of his Facebook friends.

Teacher: Morning all! Today's opening task is this; can you mark in the three numbers $\frac{1}{6}$, $\frac{1}{7}$ and $\frac{1}{8}$ on the number line?

A; I can do that roughly I think…



B: That looks about right, A.

C: But is $\frac{1}{7}$ closer to $\frac{1}{8}$ or to $\frac{1}{6}$? Or is it bang in the middle?

A: Good question. Off the top of my head, I reckon it's closer to $\frac{1}{6}$ …

D: No, I'm sure $\frac{1}{7}$ is closer to $\frac{1}{8}$ than $\frac{1}{6}$! I've worked it out.

Teacher: How did you reason, D?

D: I thought about the number 6 x 7 x 8, let's call it n. Then;

$\frac{1}{6}n = 7 \times 8 = 56$, $\frac{1}{7}n = 6 \times 8 = 48$, $\frac{1}{8}n = 6 \times 7 = 42$.

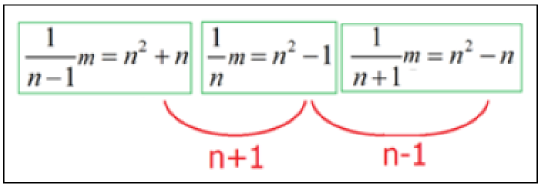
We can see $\frac{1}{7}n - \frac{1}{8}n = 6 < 8 = \frac{1}{6}n - \frac{1}{7}n$, so $\frac{1}{7}$ is closer to $\frac{1}{8}$ than $\frac{1}{6}$.

A: Or we could say $\frac{1}{6} = \frac{7 \times 8}{6 \times 7 \times 8}$, $\frac{1}{7} = \frac{6 \times 8}{6 \times 7 \times 8}$, $\frac{1}{8} = \frac{6 \times 7}{6 \times 7 \times 8}$, which goes the same way.

Teacher: Excellent. Can we generalise that?

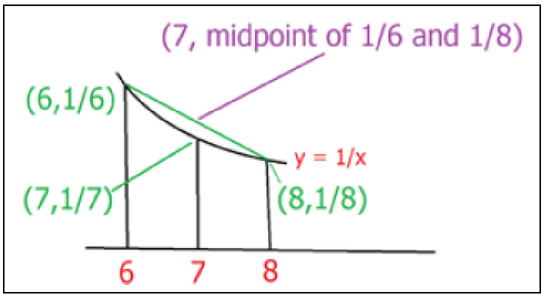
E: I can feel the word 'theorem' coming…

D: The integers 6, 7 and 8 are consecutive. So let's look at the three consecutive integers $n - 1$, n and $n + 1$. We can consider the number $(n - 1)n(n+1) = m$. Then



Our conclusion; $\frac{1}{n}$ is always closer to $\frac{1}{n+1}$ than it is to $\frac{1}{n-1}$.

B: Could we look at the graph of $y = \frac{1}{x}$ and the way it bends? It's concave - look…



The diagram shows that $\frac{1}{7}$ must be closer to $\frac{1}{8}$ than $\frac{1}{6}$ because of the way the curve bows.

E: And since the curve always bows the same way, $\frac{1}{n}$ is always going to be closer to $\frac{1}{n+1}$ than $\frac{1}{n-1}$.

C: Does that work for other concave curves? It will, surely.

D: How about $y = x^2$ for x positive? That's concave.

A: That will tell us that n^2 for n positive is always going to be closer to $(n - 1)^2$ than $(n + 1)^2$…

E: Algebra deals with that quickly, A, I think; $(n + 1)2 - n^2 = 2n + 1$, while $n^2 - (n - 1)^2 = 2n - 1$.

C: Agreed, E, but if we can say something general about any concave curve, might that save us time in the future?

Teacher: Let's change our focus slightly; consider the three fractions $\frac{1}{15}, \frac{1}{5}, \frac{1}{3}$.

Now so $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$, $\frac{1}{5} - \frac{1}{15} = \frac{2}{15}$, so $\frac{1}{5}$ is bang in the middle of the interval $\left[\frac{1}{15}, \frac{1}{3}\right]$. We could say we have a ***fraction triple*** $(15, 5, 3)$. There are going to be lots of fraction triples like this; can we find a formula that gives all of them?

A: So we need to find all integer triplets (a, b, c) where $a > b > c > 0$ such that $\frac{1}{b}$ is bang in the middle of $\left[\frac{1}{a}, \frac{1}{c}\right]$.

Teacher: I'm asking for a ***parametrisation*** of all such triples; it could be that

$(a, b, c) = (f(m, n), g(m, n), h(m, n))$

for some functions f, g and h. The variables m and n would be the two parameters here.

B: I can't help noticing that $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} \Leftrightarrow \frac{1}{kb} - \frac{1}{ka} = \frac{1}{kc} - \frac{1}{kb}$. So (a, b, c) is a fraction triple if and only if (ka, kb, kc) is.

C: So we can always reduce any triple down to (a, b, c), where a, b and c do not share a common factor.

E (who has been quiet for a while): Let me see…

$(m(m + 1), m^2 - 1, m(m - 1))$.

Is this always a one-parameter fraction triple?

D: Now (a, b, c) is a fraction triple if and only if $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$, which becomes $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$.

We can multiply that out to give us $2ac - ab - bc = 0$.

A: So that's the condition for (a, b, c) to be a fraction triple. If we substitute in for a, b and c from your suggestion, does

$2m(m + 1)m(m - 1) - m(m + 1)(m^2 - 1) - (m^2 - 1)m(m - 1) = 0$?

Multiplying out, that works!

Teacher: That's a good start, but maybe we should be careful. This formula always works, but does it catch every possibility?

C: You mean, is there a fraction triple that doesn't fit this pattern?

B: How about (15, 5, 3)? That's a fraction triple, but you can't write 3 as $m(m - 1)$.

E: Darn it! But I think I can refine my formula… is every fraction triple of the two-parameter form

$(m(m + n), m^2 - n^2, m(m - n))$, where $m > n$?

B: The fraction triple (15, 5, 3) is okay now; the values $m = 3$, $n = 2$ do the job.

A: Certainly $2ac - ab - bc = 0$ is true again, so this is always a fraction triple.

C: But does this get them all?

B: Isn't there a problem with (15, 12, 10)? If $10 = m(m - n)$ then either $m = 10$, $n = 9$, or $m = 5$, $n = 3$. Neither of these give 12 for b.

E: Darn it again! Your talent for finding counterexamples, B, is stretching me! Give me one last chance; I think I can see the problem with my latest formula. Now, does

$(m(m + n), 2mn, n(m + n))$, *where* $m > n$,

fill the gaps? I'm feeling confident now. Add this last formula on to the one I gave you before, and that will surely give the lot!

B: Certainly $2ac - ab - bc = 0$ works here.

C: The fraction triple (15, 12, 10) that B raised is of this type; $m = 3$, $n = 2$. So that's covered.

A: You're saying, E, that any fraction triple will be of the form

1. $(m(m + n), m^2 - n^2, m(m - n))$, where $m > n$, or
1. $(m(m + n), 2mn, n(m + n))$, where $m > n$,

or a multiple of one of these.

E: Yes – I stake my reputation on it!

Teacher: One day, E, you will have to share with us how you intuit these things. But we don't yet have a proof that there are no further gaps.

C: I'm thinking about the parametrisation for the primitive Pythagorean triples; aren't they

$(2mn, m^2 - n^2, m^2 + n^2)$, with $m > n$?

E: D, you've been quiet for a while! What have you been working on?

D: I've been thinking about that equation $2ac - ab - bc = 0$….

Teacher: That's the equation that all fraction triples and only fraction triples satisfy.

D: Yes; if you put $x = b$, $y = a - c$, $z = a + c - b$, then $a = \frac{x + y + z}{2}$, $b = x$, $c = \frac{x + z - y}{2}$, which means $2ac - ab - bc$ $= 2\left(\frac{x + y + z}{2}\right)\left(\frac{x + z - y}{2}\right) - \left(\frac{x + y + z}{2}\right)x - \left(\frac{x + z - y}{2}\right)x$. And if you expand that you get $-\frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2}$.

A: So $2ac - ab - bc = 0$ if and only if $x^2 + y^2 = z^2$!

B: That means (a, b, c) is a fraction triple if and only if (x, y, z) is a Pythagorean triple. Neat!

C: Doesn't that help with our parametrisation – if we use the parametrisation for Pythagorean triples? Putting $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$, we get;

$a = m(m + n)$, $b = m^2 - n^2$, $c = m(m - n)$, where $m > n$.

Whereas if we put $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$, we get;

$a = m(m + n)$, $b = 2mn$, $c = n(m + n)$, where $m > n$.

D: These match exactly with yours, E - so you got the parametrisation, no gaps, well done!

Teacher: And now we have the proof that there are no gaps that we were looking for.

A: Here's something else; (20, 15, 12) is a fraction triple, and so is (15, 12, 10). So putting these together, we get a fraction quartet, (20, 15, 12, 10).

B: So we can make fraction chains; and how long can such chains be?

C: Does an infinite chain exists?

B: I don't think an infinite chain can exist. Suppose our chain starts $(a_1, a_2, a_3, \ldots)$.

Then $a_1 > a_2 > a_3 \ldots$, all the a_i have to be positive integers, and the length of this chain will be at most a_1.

E: But could a chain be extended backwards to infinity?

B: If such a chain exists starting with b, c at the right-hand end, then if we do the algebra, it has to look like this; $\left(\ldots \frac{bc}{(n+1)c-nb}, \ldots, \frac{bc}{4c-3b}, \frac{bc}{3c-2b}, \frac{bc}{2c-b}, b, c\right)$

Now since $b > c$, eventually $(n+1)c - nb$ will be negative as n increases, which is not allowed. So no, we can't extend a chain backwards indefinitely.

C: But maybe we can find chains that are arbitrarily long?

A: Is that different from finding an infinitely long chain?

C: Yes! For an arbitrarily long chain, I'm saying, 'You give me a number n, I'll find you a chain that is at least that long.'

A: I'll give you the number n then! Can you give me a chain of length at least n?

C: I think I can. Start a chain with a and b; that means $c = \frac{ab}{2a-b}$. We can iterate that, giving the chain $\left(a, b, \frac{ab}{2a-b}, \frac{ab}{3a-2b}, \frac{ab}{4a-3b}, \ldots\right)$. So to give a chain of length n we want $\frac{ab}{(p+1)a-pb}$ to be an integer for $1 \le p \le n-2$. Now pick $a = 2(n!)$ and $b = n!$ which gives $\frac{ab}{(p+1)a-pb} = \frac{2(n!)^2}{(p+1)2(n!)-p(n!)} = \frac{2(n!)}{p+2}$.

But $p + 2$ divides into $n!$ for $1 \le p \le n-2$, so this is a whole number for those values of p. We have our chain of length at least n.

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