Getting back to where you started

A mathematical *Twitter* conversation between Oliver Gray, Jonny Griffiths and Susan Whitehouse.

Venue: a café serving delicious coffee and fresh croissants.

A: Morning B and C!

B: Hi there, A! You will doubtless have one of your nice maths questions to get us thinking today.

A: Indeed, B (pausing for bite of croissant). What do you make of this? If the 2 by 2 matrix M is such that $M^2 = I$, what are the possibilities for M, geometrically speaking?

B: We will be looking for 2 by 2 matrices M that are self-inverse.

C: Let's pick off the easy ones first. Reflection in any line through the origin will naturally do the trick. Rotation through 180° is clearly represented by such a matrix. And then there's the identity matrix itself. Is that it? It feels like it.

A: So let me throw this into the melting pot.

$$\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

B: Hmm. This is not a rotation matrix or a reflection matrix, since both require a 'cos θ' in the top left corner. There's more to this than immediately meets the eye.

C: Would you mind if we tried a bit of algebra, A? I know you've asked for a geometric interpretation, but a bit of algebra might help in that direction.

A: I did try some algebra – it's worth it! Suppose we have

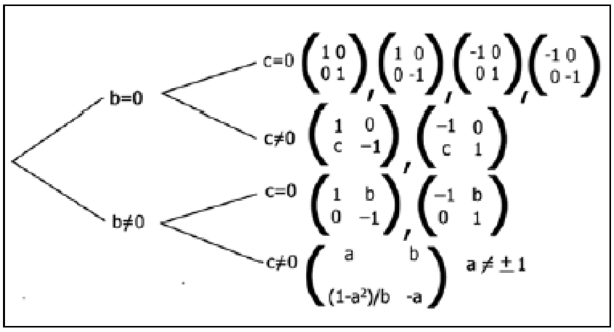
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

B: Then either $b \neq 0$, $c \neq 0$, so $d = -a$, and $a^2 + bc = 1$, which gives $c = \frac{1-a^2}{b}$, or…

C: $b = 0$, $c \neq 0$, $d = -a$ and $a^2 = d^2 = 1$, or…

A: $c = 0$, $b \neq 0$, $d = -a$ and $a^2 = d^2 = 1$, or…

C: We can have b and c being 0 together, in which case $a^2 = d^2 = 1$.



B: Great! So the top four are the identity, reflection in the y-axis, reflection in the x-axis and rotation through 180°. That's the easy ones done.

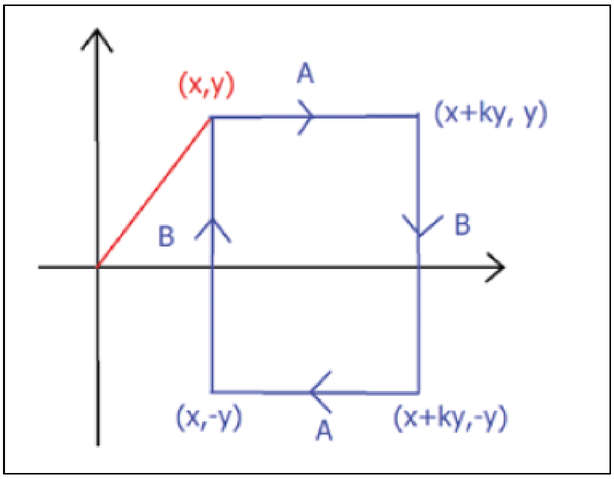
C: Putting a = cos θ and b = sin θ in the final matrix gives all the other reflections in a line through O.

A: Hmm… speaking geometrically, I think I have a handle on one of the middle matrices. Watch!

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix}$$

B: So the matrix on the right hand side here breaks down into transformation A, a shear in the x-direction, followed by transformation B, reflection in the x-axis.

C: A shearflection! What happens if we repeat one of these?



A: In the diagram (x, y) shears to (x + *k*y, y) with A, which reflects to (x + *k*y, -y) under B, which shears to (x, -y) with A, which then reflects to (x, y) with B. We

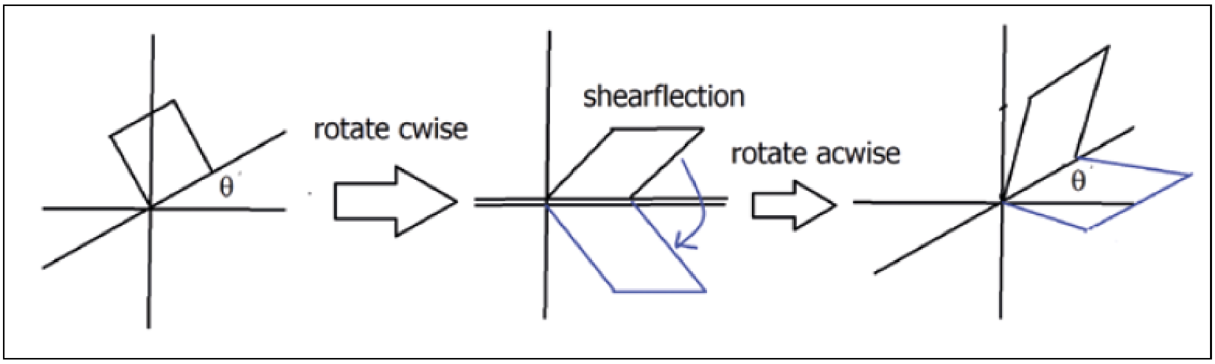


Figure 2

are back where we started - magic!

C: We could rotate the plane through an angle θ, then shearflect, then rotate back again. That would map everything to itself too when repeated.

B: Let's say $\begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix}$ is our standard shearflection, and

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is the standard rotate-shearflect-rotate-back-again transformation (call this (k, θ) form). This is only a hunch, but I reckon all of our matrices from the tree diagram can be written in (k, θ) form for some k and θ.

C: I'm not sure about that, B. The determinant of a shearflection is -1, but the determinants of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are 1.

B: Ah! Good point! But the determinants of all the other seven tree diagram matrices are all -1. I reckon all these can be written in (k, θ) form. Look at this table:

Matrix	k	-90° < θ ≤ 90°
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	0	0°
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	0	90°
$\begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$	-c	$\arctan\left(\frac{c}{2}\right)$
$\begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix}$	-c	90°
$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$	b	0°
$\begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$	b	$\arctan\left(\frac{2}{b}\right)$
$\begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}$	$\frac{a^2+b^2-1}{b}$	$\arctan\left(\frac{1-a}{b}\right)$

A: So the two parameters a and b are mirrored by the two parameters θ and k, the angle of rotation and the amount of shear. There is a bijection here between (a, b) and (k, θ), where -90° < θ ≤ 90°.

$$a = \cos 2\theta - \frac{k}{2}\sin 2\theta,\ b = \frac{k}{2}(\cos 2\theta + 1) + \sin 2\theta$$

$$\theta = \arctan\frac{1-a}{b},\ k = \frac{a^2+b^2-1}{b}$$

C: Going back to determinants, if M² = I then taking the determinant of both sides gives us (det M)² = 1, so det M = ±1.

B: Would it help to think about eigenvectors and eigenvalues?

C: Indeed. Suppose we have some matrix M so that M² = I, with two linearly independent eigenvectors **v** and **w**, with eigenvalues λ_1 and λ_2 respectively.

A: Since they are linearly independent, any vector in the plane can be written as α**v** + β**w**.

B: Now M(α**v** + β**w**) = αM**v** + βM**w** = $\alpha\lambda_1$**v** + $\beta\lambda_2$**w**.

C: That would mean

M²(α**v** + β**w**) = $\alpha\lambda_1$M**v** + $\beta\lambda_2$M**w** = $\alpha\lambda_1^2$**v** + $\beta\lambda_2^2$**w**.

A: But since M² = I, this must be α**v** + β**w**, which is only possible if $\lambda_1^2 = \lambda_2^2 = 1$.

B: That makes sense! If doing M to an eigenvector made it longer or shorter, then doing M a second time would make it even longer or even shorter, so M² could not be I.

C: Yes; the only possible eigenvalues for M are 1 and -1.

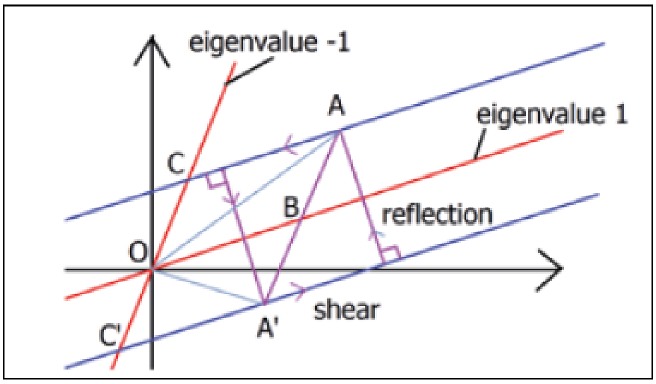
A: If we have two linearly independent eigenvectors with eigenvalue 1, well…

B: That would have to be the identity matrix. Everything is an eigenvector!

C: If we have two linearly independent eigenvectors with eigenvalue -1, then similarly…

A: The point (x, y) would go to (-x, -y) every time and that's a rotation through 180°.

B: If we have two linearly independent eigenvectors with eigenvalues 1 and -1, then can we draw a diagram?



A: So we have a general point A = (x, y) where $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$, where $\overrightarrow{OB}$ is an eigenvector with eigenvalue 1, and $\overrightarrow{OC}$ is an eigenvector with eigenvalue -1.

B: Now we do M! What happens? Point B stays where it is.

C: And point C maps to C', where OC = OC'. A goes to A' as in the diagram.

A: There's the shearflection! A shears parallel to the line OB, and then reflects in OB to get to A'.

B: Getting back to A is easy; shear parallel to OB once more, then reflect back to A in OB.

C: $M^2 = I$! So that just leaves the cases where we have a repeated eigenvalue of 1 or -1, and where the eigenvectors all lie in just one direction.

B: As for a shear in the x-direction, in fact. Although M^2 is not the identity here…

A: So the characteristic equation would have to be $(\lambda + 1)^2 = 0$ or $(\lambda - 1)^2 = 0$.

C: That's $\lambda^2 \pm 2\lambda + 1 = 0$.

B: So the determinant of M would have to be 1; but if $M^2 = I$ here, as we have seen, only the identity matrix, or the 180° rotation matrix can have determinant 1.

A: So we are done, friends; if we add on the two simple solutions we found at the start, it's shearflections all the way, rotated or otherwise.

B: Another round of croissants please!

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