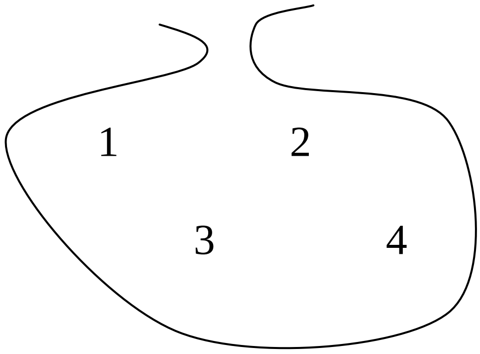
<u>Equation Solving and Hikorski Triples</u>

The following worksheet was devised initially for my GCSE Intermediate re-takers, and just went through the roof!

Put each of the numbers in the bag into a different circle, to make an equation.

$$\bigcirc\, \mathbf{x} + \bigcirc = \bigcirc\, \mathbf{x} + \bigcirc$$



How many different equations can you make?

Solve them all, and see what possible values of x arise as solutions.

Try placing your own numbers into the bag.

What is the largest possible number of different solutions for x?

Suppose that the bag contains
the distinct positive whole numbers a, b, c, d.

Let S be the set {p: p is the solution for x in one of the possible equations}

Show that if p is in S, then so is 1/p and so is -p.

What is the largest number of positive integers that S can contain?
Using your own values for a, b, c, d,
give an example of how this maximum number may be obtained.

Without doing any further calculation,
find the other members of S in this case.

If S contains three distinct positive integers,
then these are called a Hikorski triple, or an HT.

Show that putting four numbers that are in arithmetic progression into the bag will always give the same solution set S.

Show that putting four numbers in arithmetic progression into the bag will never give a Hikorski triple.

Show that putting the numbers 1, 2, 3 and 2k into the bag will always give a Hikorski triple, if k is a whole number larger than 2.

Show that the numbers 1, 2, 4 and 6k +5 will always give an HT, for k a whole number larger than 0.

Show that the positive numbers 1, 2, 5, and 12k + 10 will always give an HT, for k a whole number greater than or equal to 0.

What about 1, 2, r and s?
Can we always find a value for s that will give an HT
for any natural number r?

Compare the HT generated when 1, 2, 4, 11 are in the bag with that generated by 1, 2, 3 and 8: is this a coincidence?

Show that for any distinct natural numbers a and w,
and for any distinct natural numbers j and k greater than 1,

$$\mathbf{a,\ a + w,\ a + jw,\ a + (jk-1)(j-1)w}$$

are a quartet that give an HT when placed in the bag.

Show that the HT generated here is (jk-1-k, jk-1, (jk-1)(j-1) - j)
(which is independent of a and w).

Show that in this case, the HT can be written as:

$$x,\ \ y,\ \ \ (xy-1)/(x-y)$$

Let us say that $x \circ y = (xy-1)/(x-y)$

Show that $x \circ y$ has a curious property,
in that the recurrence relation defined by:

$$\mathbf{u_{n+2} = u_n \circ u_{n+1}}$$

is periodic, with period six.

Show that if p, q and r are an HT, $p < q < r$, then:

$$\mathbf{r \circ p = q, \quad q \circ p = r, \quad -r \circ q = p}$$

Show if α and β are two members of an HT, with $\alpha < \beta$,
then the other is either $\beta \circ \alpha$ or $(-\beta) \circ \alpha$.

Show that by putting $u_1 = q$, and $u_2 = p$,
the recurrence relation will generate half of S.

Show that putting $u_1 = (1/q)$, and $u_2 = p$,
we generate four more elements of S.

Show the rest of S is generated by $u_1 = q$, $u_2 = (1/p)$

Show that if the bag contains a, b, c, and d, then replacing them with
a + w, b + w, c + w, d + w will give exactly the same solution set S.

So therefore we need only consider the numbers 0, a, b, c in the bag
without loss of generality.

Show that if S contains an HT, then it must be possible to write it as
p, q, $q \circ p$, that is, every HT is of this form.

Show that if you have three natural numbers such that they can be written
p, q, $q \circ p$,
then these will be generated in S if the bag contains either
$\{0, q-1, p-1, pq-1\}$ or $\{0, pq-1, pq + q, pq + p\}$

Thus three distinct natural numbers p, q, r,
with $p < q < r$, are an HT $\Leftrightarrow$ $r = q \circ p$

Show that our formula (jk-1-k, jk-1, (jk-1)(j-1) - j) (call this Formula One) fails to generate all HTs, by finding one that doesn't fit the pattern.

Can we find a formula that will generate all HTs, in the way that ($2pq$, $p^2 - q^2$, $p^2 + q^2$) generates all Pythagorean triples?

Show that (jk +1 - k, jk + 1, (jk + 1) (j-1) + j) is an HT too.
(Call this Formula Two).

Find an HT that Formula Two generates,
but which Formula One does not.
(You might want to draw up a spreadsheet that shows where the HTs occur.)

Show that Formula One and Formula Two together still fail to generate all HTs.

Show that (6n + 5, 10 n + 9, 15n + 11) and (6n + 7, 10n +11, 15n + 19) give HTs for all natural numbers n,
and that these supply some of the missing HTs.

To find: a Formula that generates all HTs and only HTs!

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