Hikorski Triples

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November 2013

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Abstract

A new triple of natural numbers is introduced, $(a, b, \frac{ab+1}{a+b})$*. These* ***Hikorski Triples*** *(or HTs) arise from a simple yet evocative mathematical situation, and there is a link with the world of* ***periodic recurrence relations*** *(or* ***Lyness cycles****). When the terms of such cycles are multiplied or added, they can give rise to* ***elliptic curves****.*

Mathematics contains many positive integer triples. The most famous are the Pythagorean Triples (where (a, b, c) satisfies $a^2 + b^2 = c^2$, for example, $(3, 4, 5)$), but there are Amicable Triples (where (a, b, c) satisfies $\sigma(a) = \sigma(b) = \sigma(c) = a + b + c$, where $\sigma(x)$ is the sum of the positive divisors of x including x (an example is (1,980, 2,016, 2,556)), Markov Number Triples (where (a, b, c) satisfies $a^2 + b^2 + c^2 = 3abc$, for example, $(5, 13, 194)$), triples that arise from the Euler Brick problem ((a, b, c) where $a^2 + b^2, b^2 + c^2$ and $c^2 + a^2$ are all squares, for example,(44, 117, 240)) and so on. These examples of triples each arise from a simple mathematical situation; indeed, you might argue that the simpler the motivating situation, the profounder the significance of the triple.

Allow me to define here a new triple of positive integers, called the Hikorski Triple (hereafter referred to as an HT). The name is explained in [1] on page 200. A triple of natural numbers (a, b, c), with $a \geqslant b \geqslant c$, is an HT if and only if $c = \frac{ab+1}{a+b}$. HTs arise naturally from the following task, discussed back in *Mathematical Spectrum*, Volume 35 Number 3, June 2002/3. This simple situation arose out of writing a GCSE worksheet on equations for my students; ten years later, it provided all the raw material for my MSc by Research.

Put the numbers in the bag into the circles
in any way you choose (no repeats!) Solve the resulting equation.
What solutions are possible as you choose different orders?
What happens if you vary the numbers in the bag,
keeping them as distinct integers?

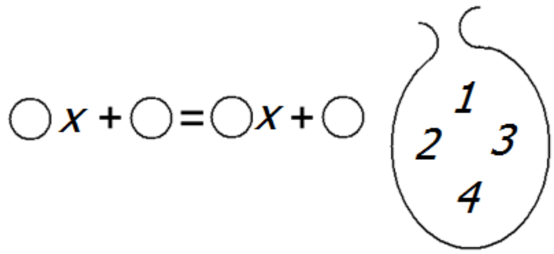

Figure 1: Numbers in a bag

The reader is invited to:

1. show that if S is the set of solutions for a given quartet in the bag, the maximum size for S is 12,

2. show that if x is in S, then $-x$, $\frac{1}{x}$, and $-\frac{1}{x}$ are also in S,

3. show that S can contain at most three positive integers, and find an example of this.

Let us now show that if S does contain three positive integers, they form an HT. The solution to $ax+b=cx+d$ is $x=\frac{d-b}{a-c}$. Thus if we replace a,b,c,d in the bag with $a+k, b+k, c+k, d+k$, we will get the same solutions. Similarly replacing a,b,c,d with $ka, kb, kc, kd (k \neq 0)$ gives the same solutions. Thus without loss of generality, we can replace our numbers in the bag with $0<1<i<j$, where i and j are rational without changing the solutions. Suppose here $j-i<1$. Then $0,-1,-i,-j$ give the same solutions, as do $j, j-1, j-i, 0$ (adding j), as do $\frac{j}{j-i}, \frac{j-1}{j-i}, 1, 0$, (dividing by $j-i$). Now note that $\frac{j}{j-i}-\frac{j-1}{j-i}=\frac{1}{j-i}>1$. So we can also insist without loss of generality that the numbers in our bag are $0<1<i<j$ where $j-i>1$. So the twelve solutions possible with this standardised bag are $\frac{j}{i-1}, \frac{-j}{i-1}, \frac{i-1}{j}, \frac{1-i}{j}, \frac{j-i}{1}, \frac{i-j}{1}, \frac{1}{j-i}, \frac{1}{i-j}, \frac{j-1}{i}, \frac{1-j}{i}, \frac{i}{j-1}$ and $\frac{i}{1-j}$. Of these, only three can be positive integers, $p=\frac{j}{i-1}, q=j-i$, and $r=\frac{j-1}{i}$. Eliminating i and j gives us $r=\frac{pq+1}{p+q}$. Notice that $(n,1,1)$ is always an HT for n a positive integer, as given by the bag $\{2n, n+1, n-1, 0\}$, for example, and is called a *trivial HT*. So in general there are twelve solutions to our numbers-in-the-bag equation, which fall into three quartets,

$$\left\{p,-p,\frac{1}{p},-\frac{1}{p}\right\},\left\{q,-q,\frac{1}{q},-\frac{1}{q}\right\},\left\{\frac{pq+1}{p+q},-\frac{pq+1}{p+q},\frac{p+q}{pq+1},-\frac{p+q}{pq+1}\right\}.$$

What resonances do the four functions in the third quartet have for us? We might think of the functions tanh and coth:

$$\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{\tanh(x)\tanh(y)+1}, \coth(x+y) = \frac{\coth(x)\coth(y)+1}{\coth(x)+\coth(y)}.$$

Relativity comes to mind. If nothing can travel faster than the speed of light, how do we add speeds? For example, if someone is moving at three-quarters the speed of light down a train that is itself moving at three-quarters the speed of light, what is the person's resultant speed? If we say that the speed of light is 1, we can add parallel speeds relativistically as follows: define $\odot$ as

$$a \odot b = \frac{a+b}{1+ab}$$

and thus $\frac{3}{4} \odot \frac{3}{4} = \frac{24}{25}$. The reader is invited to check that the set of numbers in the interval $(-1, 1)$ in $\mathbb{R}$ together with the binary operation $\odot$ form a group. Also

1. show that if in Figure 2 the lines PQ and RS are horizontal and TU is vertical, then the line AB is horizontal and of length $\frac{xy+1}{x+y}$,

2. explore a link with the world of recurrence relations. If we write the recurrence relation $u_{n+1} = f(u_{n-1}, u_n)$ with $u_0 = x, u_1 = y$ as $x, y, f(x, y), \ldots$, then show that, if $x + y$ is non-zero and $x \neq \pm 1, y \neq \pm 1$, the recurrence $x, y, -\frac{xy+1}{x+y}, \ldots$ is periodic, period 3.

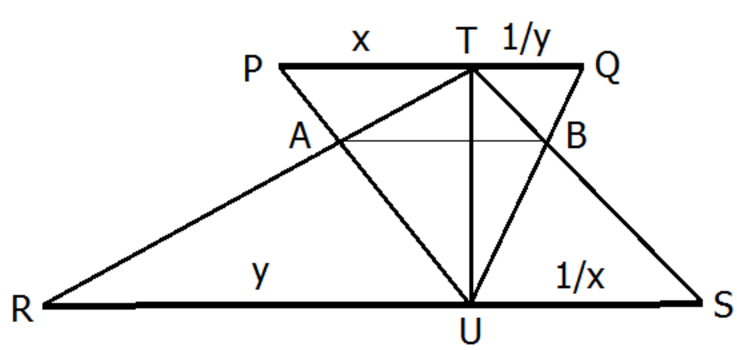

Figure 2: Constructing $\frac{xy+1}{x+y}$

How many HTs are there? They are relatively numerous; discounting the trivial HTs, there are 2252 HTs with all terms less than 1000 (the same count for Pythagorean Triples, including both primitive and non-primitive triples, yields 1034). A question arises; given n, in how many HTs does n appear? Counting up, we find that 11 features in 16 distinct HTs, while the number 12 appears in only four. Let $h(n) =$'number of HTs in which n is an element', and note that

$$(n,1,1),(n^2-n-1,n,n-1),(n^2+n-1,n+1,n),(2n+1,2n-1,n)$$

are always distinct HTs for $n > 2$. So we have $h(1) = \infty, h(2) = 2$ (2 only occurs in $(2,1,1)$ and $(5,3,2)$), and $h(n) \geqslant 4$ for $n \geqslant 3$. Now $11^2 - 1 = 120$, which has 16 positive divisors, while $12^2 - 1 = 143 = 11 \times 13$, which has only 4 positive divisors. It is easy to conjecture that $h(n) = d(n^2 - 1)$, where $d(n)$ = 'the number of positive divisors of n'.

Theorem: *The number of HTs in which n appears is $h(n) = d(n^2 - 1)$ for $n > 1$.*

Proof: The theorem is clearly true for $n = 2$. Now we have that $d(k)$ is odd if and only if k is a square, and $n^2 - 1$ cannot be a square if $n > 1$, so $d(n^2-1) = 2j$ for some j. So there exist j pairs of integers (u, v) with $u > v$, so that $uv = n^2 - 1$, where $j \geqslant 2$ since $1, n-1, n+1$ and $n^2 - 1$ all divide $n^2 - 1$ for $n > 2$. How many HTs contain n? Firstly, it is easy to show n cannot appear twice in an HT unless $n = 1$. Now consider the case where n is the smallest member of the HT $(n+u, n+v, n)$ where $u > v > 0$. We have

$$\frac{(n+u)(n+v)+1}{2n+u+v} = n \Leftrightarrow uv = n^2 - 1.$$

There are exactly j pairs for (u, v) that satisfy this equation, so there are j HTs with n as the smallest member. Now consider the case where n is not the smallest member of the HT, so we may write the HT as $(u-n, n, n-v)$, where $n - v$ is the smallest element (we cannot be sure how $u - n$ and n are ordered), with $1 \leqslant v < n, u > n$. We have from the HT definition

$$\frac{(u-n)n+1}{u} = n - v \Leftrightarrow uv = n^2 - 1.$$

We know there are precisely j values for v between (and including) 1 and $n-1$ that divide n^2-1, and each gives a value for u between (and including) $n+1$ and n^2-1 that divides n^2-1, and so $u-n$ is between (and including)

1 and n^2-n-1. We need to check that $u-n \geqslant n-v$ in every case, which is true if and only if $u+v \geqslant 2n$. But $u+v \geqslant 2\sqrt{uv}$ by the AM-GM inequality, and so $u+v \geqslant 2\sqrt{n^2-1}$. Now

$$2n > 2\sqrt{n^2-1} > 2n-1 \Leftrightarrow 4n^2 > 4n^2-4 > 4n^2-4n+1$$

which is true, and so since $u+v$ is an integer, $u+v \geqslant 2n$ for $n>1$. Thus we always have exactly $2j$ HTs for a given $n>1$, and so $h(n)=d(n^2-1)$.

It is worth noting that if $n-1$ and $n+1$ are both prime (that is, they form a prime pair), then n^2-1 has four positive divisors, $1, n-1, n+1$ and $(n-1)(n+1)$, so that $h(n^2-1)=4$. Are there infinitely many n so that $h(n)=4$?

The Uniqueness Conjecture for HTs

If we add the elements of an HT, the sum may well be shared with other HTs. Indeed, we can prove that the number of HTs whose three elements add to k is unbounded as $k \to \infty$. Multiplying the elements of an HT, however, proves to be more intriguing. Of course the trivial HTs $(n,1,1)$ can multiply to any whole number. What if we exclude these? A computer program enables a reasonably swift check on the first 250 000 non-trivial HTs, and multiplying their elements gives distinct products. The first few HTs with their products are given in Table 1. And so we have the following uniqueness

a	5	7	11	9	19	11	13	17	13	29	15
b	3	5	4	7	5	9	8	7	11	6	13
c	2	3	3	4	4	5	5	5	6	5	7
abc	30	105	132	252	380	495	520	595	858	870	1365

Table 1: Early HT Products

conjecture (called from now on the UC) that remains open as I type, and which is offered as a challenge to the reader.

Conjecture: *If (a,b,c) and (p,q,r) are non-trivial HTs with $abc=pqr$, then $(a,b,c)=(p,q,r)$.*
Here are two HTs (a,b,c) and (p,q,r) where abc is close to pqr – a counterexample to the idea that such products are necessarily far apart:

(1957, 1955, 978) is an HT, and $1957 \times 1955 \times 978 = 3,741,764,430$, while (7897, 719, 659) is also an HT, and $7897 \times 719 \times 659 = 3,741,764,437$.

The UC initially looks straightforward to prove, but if it holds, the consensus of experienced observers is that a proof may well be difficult. There is coincidentally also a uniqueness conjecture for Markov Triples (see [1] page 175), which also remains unresolved, despite much attention.

The $-\frac{xy+1}{x+y}$ cycle

Given our interest in HTs, the period-3 recurrence

$$x, y, -\frac{xy+1}{x+y}, x, y \ldots.$$

deserves special consideration. (A recurrence that is periodic in general for any starting terms is known as a *Lyness cycle*. R. C. Lyness was a mathematician and school-teacher who first discussed such recurrences - see [1] page 22.) Multiplying the three terms and equating to $-k$ gives the curve

$$xy(-\frac{xy+1}{x+y}) = -k, \quad \text{or} \quad x^2y^2 + xy - kx - ky = 0. \tag{1}$$

This is an example of an *elliptic curve*, which is essentially a curve of the form $y^2 = P(x)$ where $P(x)$ is a polynomial of degree 3 that does not factorise, and which has no singularities, that is, crossing points or cusps. Some quartic curves reduce to elliptic ones. Using the transformation

$$x = \frac{X}{4k}, y = \frac{2kY - 2kX + 8k^3}{X^2}, X = 4kx, Y = 8kx^2y + 4kx - 4k^2,$$

our curve becomes

$$Y^2 = X^3 + X^2 - 8k^2X + 16k^4.$$

This is now in *Weierstrass form*, a standard representation for elliptic curves.

Integer point implications

Elliptic curves are vital in many areas of mathematics. When Hardy and Wright's masterpiece *An Introduction to the Theory of Numbers* reached its sixth edition, just a single chapter was deemed to be an essential addition to the original, one on elliptic curves. One of the many remarkable properties of these curves is that the points they contain form a group under a certain simple addition law. They also form the basis for much of our most effective cryptography. Elliptic curve theorems include Siegel's, proved in 1929, which tells us there are only finitely many integer points on any elliptic curve. What, then, will be the integer points on (1) if k is $30 = 5 \times 3 \times 2$? (The triple (5, 3, 2) is an HT.) First note that given the periodic nature of our recurrence, the expression $xy(-\frac{xy+1}{x+y})$ is invariant under the transformation $T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -\frac{xy+1}{x+y} \end{pmatrix}$. Since (5, 3, 2) is an HT, integer points will therefore include (5, 3), (3, -2) and (-2, 5). Their reflections in $y = x$, (3, 5), (-2, 3) and (5, -2) can also be added. We also have (30, 1), (1, -1), (-1, 30) , (30, -1), (-1, 1) and (1, 30) from the trivial HT, a total of 12 integer points. But if the UC is true, this will be all, and we will therefore have a family of elliptic curves (as we vary k to give HT products) that each have 12 integer points (that are easily found), and only 12. The unlikelihood of this situation casts doubt on the plausibility of the UC, but if the UC turns out to be true, then the implications are exciting.

Bibliography

[1] J. Griffiths. *Lyness Cycles, Hikorski Triples and Elliptic Curves* MSc Thesis, University of East Anglia, 2012, **http://www.s253053503.websitehome.co.uk/jg-msc-uea/index.html** (accessed November 2013).

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