Lyness Cycles and their Invariant Curves

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Abstract

Second-order recurrence relations are often associated with invariant curves. Although they are not particularly common, some order-2 recurrences exhibit periodic behaviour, as their terms repeat, and such recurrences possess invariant curves that are relatively easy to find. In this article we explore some of the properties of these recurrence relations; in particular, we examine the way in which the invariant curves for cycles of different periods can almost coincide, as they degenerate into three straight lines.

Introduction

Readers may well be familiar with the idea of a recurrence relation, and how this may be used to define infinite sequences. For example, the simple recurrence

$$u_{n+2} = u_{n+1} + u_n, u_1 = 1, u_2 = 1$$

defines the Fibonacci sequence 1, 1, 2, 3, 5, 8, ... We call this an order-2 recurrence, as the next term is defined in terms of the previous two. Now consider the mapping

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x+y \end{pmatrix}.$$

If we put x and y equal to 1, and iterate T, it is easy to see how the Fibonacci sequence is generated (in the more general case, we can choose any starting values for x and y to produce a Fibonacci-type sequence).

Now consider the curve $(x^2 + xy - y^2)^2 = k^2$ (which could be regarded as the union of two conics). Let's choose a point $\begin{pmatrix} a \\ b \end{pmatrix}$ and choose k so that this point is on the curve. Now $T\begin{pmatrix} a \\ b \end{pmatrix}$ is also on the curve if and only if $(b^2 + b(a+b) - (a+b)^2)^2 = k^2$, or $(b^2 + ab + b^2 - a^2 - 2ab - b^2)^2 = k^2$, or $(a^2 + ab - b^2)^2 = k^2$, which is true. So clearly $T^n\begin{pmatrix} a \\ b \end{pmatrix}$ will be on the curve for all n, and this will be true for any starting point on the curve (see Figure 3). We say $(x^2 + xy - y^2)^2 = k^2$ is an invariant curve for the recurrence; as T is iterated, the points generated appear on each conic in turn.

The reader might justifiably point out that this curve appeared out of nowhere, and we would have to agree – invariant curves can be hard to find. But in certain rare cases, the order-2 recurrence of interest is periodic (its terms repeat in a cycle), and in such circumstances it's possible to find invariant curves with relative ease. These curves in a profound sense share the period of the generating recurrence – if the recurrence is period-5, then the invariant curve is period-5 too.

The invariant curves here are often elliptic (that is, cubic curves without singular points, like cusps or crossings), but they can sometimes become degenerate (cubic curves WITH singular points, for example, if the equation factorises). Our aim here is to show how, in the situation for which the equations for a set of invariant curves come close to factorising into a degenerate set of three straight lines, it's possible for invariant curves of different periodicities to become as close to each other as we like.

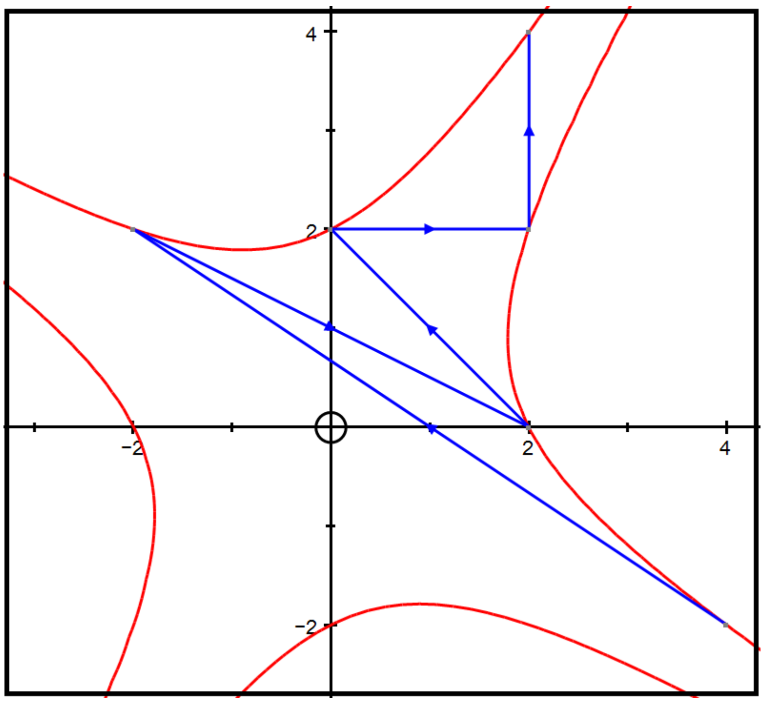

Figure 1: Fibonacci-type sequence $4, -2, 2, 0, 2, 2, \ldots$ with invariant curve $(x^2 + xy - y^2)^2 = 4^2$

An interesting recurrence

Consider the order-1 recurrence relation

$$u_{n+1} = \frac{3u_n + 5}{1 - u_n}.$$

If we put $u_1 = x$, then we can find subsequent terms in terms of x, and something pleasing happens. The sequence goes

$$x, \frac{3x+5}{1-x}, -\frac{x+5}{x+1}, \frac{x-5}{x+3}, x, \ldots. \tag{1}$$

So if $f(x) = \frac{3x+5}{1-x}$, then $f^4(x) = x$, and the recurrence returns to its start; we say it is period-4.

We note here that $f(x)$ is periodic not just for some freak value of x, but for all values of x except for the special cases $x = 1$, $x = -1$ and $x = -3$. This globally periodic behaviour is unusual among recurrence relations.

What periods are possible in the order-1 case?

A natural next question: If $f(x) = \frac{ax+b}{cx+d}$, where a, b, c and d are rational, what periods are possible for the recurrence $x, f(x), f^2(x), \ldots$? There is a lot of drudgery if we tackle this by hand, but a computer algebra package makes things much easier. It turns out that the only possible periods are 1, 2, 3, 4 and 6. Cull, Flahive and Robson [2] give a proof that is accessible, but far from trivial.

At its heart is *Euler's totient function* $\phi(n)$ [1], the number of positive integers smaller than n that are coprime with n. For example, $\phi(8) = 4$, $\phi(9) = 6$ and $\phi(p) = p - 1$ if p is prime. We may note also that $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, and $\phi(6) = 2$. It is also true that $\phi(n) \geqslant 3$ for $n > 6$. The proof mentioned above shows that a period-n cycle is impossible if $\phi(n) \geqslant 3$, which is why 5 misses out.

A related quartic equation

Multiplying the terms in the non-repeating block in (1) together and equating this to a constant k turns out to be helpful idea. We obtain

$$x\left(\frac{3x+5}{1-x}\right)\left(-\frac{x+5}{x+1}\right)\left(\frac{x-5}{x+3}\right) = k.$$

This is a quartic equation. What happens if we substitute in $\frac{3x+5}{1-x}$ for x? We get

$$\left(\frac{3x+5}{1-x}\right)\left(\frac{3(\frac{3x+5}{1-x})+5}{1-(\frac{3x+5}{1-x})}\right)\left(-\frac{(\frac{3x+5}{1-x})+5}{(\frac{3x+5}{1-x})+1}\right)\left(\frac{(\frac{3x+5}{1-x})-5}{(\frac{3x+5}{1-x})+3}\right) = k,$$

which, by periodicity, is

$$\left(\frac{3x+5}{1-x}\right)\left(-\frac{x+5}{x+1}\right)\left(\frac{x-5}{x+3}\right)x = k.$$

The equation is thus left unchanged by this substitution (it is invariant). It follows from this that if we know that one root is α, the other roots are given by $\frac{3\alpha+5}{1-\alpha}$, $-\frac{\alpha+5}{\alpha+1}$ and $\frac{\alpha-5}{\alpha+3}$.

The periodic order-2 recurrence

We have investigated an order-1 example; what happens for order-2 recurrences, like our Fibonacci example? Are there periodic examples here, and if so, which periods are possible? We turn now to one of the most famous examples in the literature:

$$u_{n+2} = \frac{u_{n+1} + 1}{u_n}$$

for $n \geq 1$, where $u_1 = x$ and $u_2 = y$.

Iterating, we get the sequence

$$x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y}, x, y, \ldots. \qquad (2)$$

This is known as a *Lyness cycle*, [4, 5, 6], after Robert Cranford Lyness, the late mathematics teacher, inspector and Olympiad organiser. The way the algebra simplifies halfway through is a joy. Again, this is globally periodic, as long as x and y are such that we never divide by zero.

What periods are possible in the order-2 case?

So what periods are possible here? If $f(x,y) = \frac{p(x,y)}{q(x,y)}$, where p and q are polynomials with rational coefficients, what periods are possible for $x, y, f(x,y), \ldots$? Certainly periods 2, 3, 4, 5, 6, 8, and 12 are possible. We have

1. period-2. $x, y, x, y, \ldots$
2. period-3. $x, y, \frac{1}{xy}, x, y, \ldots$
3. period-4. $x, y, -\frac{(x+1)(y+1)}{y}, -\frac{xy+x+1}{x+1}, x, y, \ldots$
4. period-5. $x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y}, x, y, \ldots.$
5. period-6. $x, y, \frac{y}{x}, \frac{1}{x}, \frac{1}{y}, \frac{x}{y}, x, y, \ldots$
6. period-8. $x, y, \frac{x-1}{x+1}, \frac{y-1}{y+1}, -\frac{1}{x}, -\frac{1}{y}, \frac{x+1}{1-x}, \frac{y+1}{1-y}, x, y, \ldots$
7. period-12. $x, y, \frac{2x-1}{x+1}, \frac{2y-1}{y+1}, \frac{x-1}{x}, \frac{y-1}{y}, \frac{x-2}{2x-1}, \frac{y-2}{2y-1}, \frac{1}{1-x}, \frac{1}{1-y}, \frac{x+1}{2-x}, \frac{y+1}{2-y}, x, y, \ldots$

as long as once again, x and y are such that we never divide by zero. The examples for periods 8 and 12 might, however, be regarded as a little unsatisfactory; they are formed by interleaving two order-1 recurrences (we call these *pseudo-cycles*). Tsuda [7] has shown that cycles of the form

$$x, y, \frac{f(y) - g(y)x}{g(y) - h(y)x}, \ldots.$$

where f, g, and h are rational polynomials of degree at most 4, can only possess the periods 2, 3, 4, 5 and 6. (If we allow θ to be real, the recurrence $x, y, \theta y - x, \ldots$ can have any period for the right choice of θ [3].)

An invariant curve for the Lyness cycle

Now we return to (2) What happens if we multiply the terms of the recurrence together and equate the result to a constant k? We obtain

$$xy\left(\frac{y+1}{x}\right)\left(\frac{x+y+1}{xy}\right)\left(\frac{x+1}{y}\right) = k.$$

This gives us the cubic equation $(x+1)(y+1)(x+y+1) = kxy$, an invariant curve for this Lyness cycle. Using the same logic used in the order-1 example, if we know that (x, y) is on the curve, then, by periodicity, (x, y), $\left(y, \frac{y+1}{x}\right)$, $\left(\frac{y+1}{x}, \frac{x+y+1}{xy}\right)$, $\left(\frac{x+y+1}{xy}, \frac{x+1}{y}\right)$ and $\left(\frac{x+1}{y}, x\right)$ will also be on the curve. We can say the curve is globally period-5; this time starting from (almost) any point on the curve, the terms form a closed circuit(see Figure 2).

Note also that we could arrive at an invariant curve for the cycle using any symmetric function in the terms; for example, if the cycle is period 5, and the terms are $x, y, p, q, r, x, y, \ldots$, then

$$xypqr = k, x+y+p+q+r = k, xy+yp+pq+qr+rx = k,$$

and so on, will all be invariant curves for the recurrence.

Different periods, but (almost) the same invariant curve

Now to the observation that we hope is original. We ask the following question: is it possible for an invariant curve to host cycles of different periods? The immediate answer would seem to be "No"; if a curve is globally periodic period-5, then how can there be a point on the curve that is period-6? (For elliptic curve specialists, the period of the recurrence must match the torsion of the curve). There is only one way out of this impasse; the invariant curve must collapse into three straight lines (become degenerate), and now, when it comes to periodicity, we might say that all bets are off.

So consider the Lyness cycles below:

1. period-3; $x, y, \frac{2xy-22x-22y+62}{xy-2x-2y+22}, x, y, \ldots$

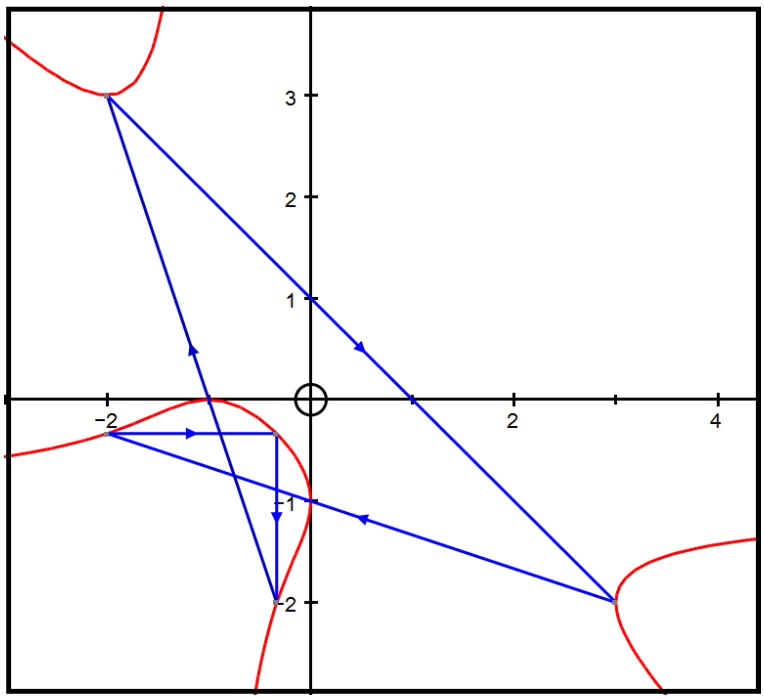

Figure 2: $(x+1)(y+1)(x+y+1) = kxy$ for $k = 4/3$ starting from (3,-2)

2. period-4; $x, y, \frac{xy-y^2+y-2}{xy-x-y}, \frac{x^2-xy+x+2}{y-xy+x}, x, y, \ldots$

3. period-5; $x, y, \frac{5-x-3y}{x+1}, \frac{-xy+8x+8y-10}{(x+1)(y+1)}, \frac{5-3x-y}{y+1}, x, y, \ldots$

4. period-6; $x, y, \frac{x-3y+1}{2x-1}, \frac{x+4}{2x-1}, \frac{y+4}{2y-1}, \frac{3x-y-1}{1-2y}, x, y, \ldots$

In each case adding the terms in a block and equating to k gives an invariant curve for the cycle. If we then put k equal to 3 (in the period-3 case), 5 (in the period-4 case), 4 (in the period-5 case) and 6 (in the period-6 case) then, in each instance, the curve (remarkably) simplifies to $(x-2)(y-2)(x+y-1) = 0$, or three straight lines (checking this DEFINITELY needs a computer algebra system!).

Conclusion

Maybe this is nothing more than a curiosity. But it does show that we can find invariant curves for Lyness cycles which possess different periods, and yet are arbitrarily close to each other (see Figure 3). We hope that this fact is worth noting.

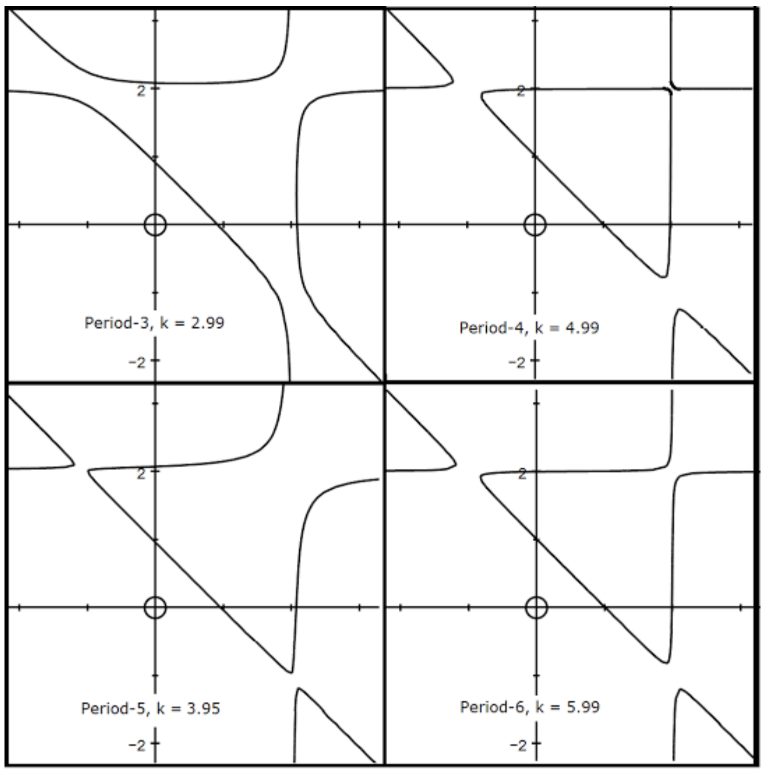

Figure 3: Invariant curves of period 3, 4, 5 and 6 that approximate to three straight lines

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The authors are not related(!)