Lopsided Numbers

I was discussing inversion with my students the other day, the transformation that takes (r, θ) to $(\frac{1}{r}, \theta)$ in polar coordinates.

"Think of these points as being best friends," I said whimsically, "On the understanding that your best friend's best friend will be yourself."

"Aren't there such things as *friendly* numbers?" asked Daniel, a bright student who reads around the subject. The number 284 came into my mind, and a visit to Wikipedia revealed why – 284 and 220 are *amicable* numbers.

"What does that mean?" Tilly asked.

"Define σ(n) as the sum of all the divisors of n, including 1 and n, and σ'(n) as the sum of all the proper divisors of n, that is, not including n," I replied. "Now σ'(284) = 220, while σ'(220) = 284."
The website Wikipedia (see reference 1) obligingly told us further that a friendly number was something different. $\frac{\sigma(n)}{n}$ takes a range of values as n varies. If $\frac{\sigma(m)}{m} = \frac{\sigma(n)}{n}$, then n and m are friends, and are thus both friendly numbers. If $\frac{\sigma(n)}{n}$ for some n takes a value that is unique, then n is a *solitary* number. It turns out that all primes are solitary (proving this is included in this issue's Problems section).

[It is worthwhile mentioning that σ(n) is a *multiplicative* function – that is, if n and m have no common factor, then σ(nm) = σ(n) σ(m).]

Alongside this information, Wikipedia helpfully offered a large box of further definitions: *hyper-perfect* numbers, *practical* numbers, *weird* numbers, *sublime* numbers, *frugal* numbers and so on. This much was immediately clear - some hypotheses that are extremely simple to state remain unresolved in this area. Is there an odd *perfect* number (a perfect number is one where σ'(n) = n, for example 6, 28, 496…)? Is 10 a solitary number? How many sublime numbers are there? (A sublime number is one which has a perfect number of positive divisors (including itself), and whose positive divisors add up to another perfect number. The number 12, for example, is a sublime number. It has a perfect number of positive divisors (6): 1, 2, 3, 4, 6, and 12, and the sum of these is again a perfect number: 1 + 2 + 3 + 4 + 6 + 12 = 28. There are only two known sublime numbers, 12 and

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Readers are invited to check this!)

Tilly had been thinking. "σ(n) is always bigger than n, while σ'(n) can be less than, or equal to, or greater than n," she said. "Can you find a pair of numbers n and m, with n less than m, so that:

σ(n) = m and σ'(m) = n?"

We looked at this for a moment, and couldn't see any good reason why such a pair of numbers should not exist. Pete, my Head of Department scratched his head when presented with the problem later; "They would be *lopsided* numbers, then."

So we seek a pair, $(n, \sigma(n))$ so that $\sigma'(\sigma(n)) = n$. Playing with small numbers, we found that such pairs are certainly possible.

$\sigma(3) = 4$ and $\sigma'(4) = 3$, so $(3, 4)$ is a lopsided pair.

$\sigma(7) = 8$ and $\sigma'(8) = 7$, so $(7, 8)$ is a lopsided pair.

Is $(2^k-1, 2^k)$ always a lopsided pair? No, since $\sigma(15) = 24$ and $\sigma'(24) = 36$.

What if $2^k -1$ is prime? Now $\sigma(2^k-1) = 2^k$ and $\sigma'(2^k) = 1 + 2 + 4 \ldots + 2^{k-1} = 2^k-1$.

So $(2^k-1, 2^k)$ is a lopsided pair if 2^k-1 is prime (such primes are called *Mersenne* primes).

Our question now becomes, are there any lopsided pairs that are not of this form? At this point, a simple Excel program provides us with a lot of help. It reveals no lopsided pairs (n, m) that are not of the above form (for n up to50 000.) It does, however, supply us with some near misses.

$\sigma(18) = 39$ and $\sigma'(39) = 17$, so $(18, 39)$ is almost a lopsided pair.

$\sigma(242) = 399$ and $\sigma'(399) = 241$, so $(242, 399)$ is almost a lopsided pair.

$\sigma'(94) = 50$ and $\sigma(50) = 93$, so $(50, 94)$ is almost a lopsided pair.

$\sigma'(2457) = 2023$ and $\sigma(2023) = 2456$, so $(2023, 2457)$ is almost a lopsided pair.

Notice there is a 'falling one short' each time – it makes sense to call these 'almost lopsided numbers', since 'almost perfect numbers' are those where $\sigma(n) = 2n - 1$.

If $\sigma(n) = 2n + 1$, then n is called *quasiperfect*, but no such numbers have been discovered thus far. My Excel program revealed no 'quasilopsided' pairs for n under 50 000. Why should it be so much easier to fall short by 1 rather than to overreach by 1?

So our unproven conjecture remains as follows;

(n, m) is a lopsided pair $\Leftrightarrow$ (n, m) is of the form $(2^k-1, 2^k)$ where 2^k-1 is prime.

This may be no easier to prove than any of the other conjectures in the field, but MS readers are invited to try.

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Reference

1. Wikipedia - http://en.wikipedia.org

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