Dear Editor,

Periodic recurrence relations

I enjoyed Jonny Griffiths' article on *Periodic recurrence relations of the type* $x, y, y^k/x, \ldots$ in *Mathematical Spectrum*, Volume 37, Number 2. A slight shift in perspective from algebra towards trigonometry enables many of the loose ends to be tied up.

The recurrence relation $v_n = kv_{n-1} - v_{n-2}$, with $v_0 = 0$ and $v_1 = 1$, has auxiliary equation $\lambda^2 - k\lambda + 1 = 0$ with roots $k/2 \pm \sqrt{k^2/4 - 1}$. Solving this in the usual way gives the following solutions.

Case 1 If $k = 2$ then $v_n = n$, and if $k = -2$ then $v_n = n(-1)^{n+1}$.

Case 2 If $|k| > 2$ then

$$v_n = \frac{1}{\sqrt{k^2-4}}\left[\left(\frac{k}{2} + \sqrt{\frac{k^2}{4} - 1}\right)^n - \left(\frac{k}{2} - \sqrt{\frac{k^2}{4} - 1}\right)^n\right].$$

(Thus, if $|k| \geq 2$ then (v_n) is unbounded and hence divergent.)

Case 3 If $|k| < 2$ then $k = 2\cos\theta$ for some $0 < \theta < \pi$, and the roots of the auxiliary equation are $\cos\theta \pm \mathrm{i}\sin\theta$, so that $v_n = \sin n\theta / \sin\theta$.

Note that the formula

$$v_n = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-1-r}{r} k^{n-2r-1}$$

quoted in Griffiths' article thus corresponds to the following trigonometric identity:

$$\sin n\theta = \sin\theta \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-r-1}{r} 2^{n-2r-1} \cos^{n-2r-1}\theta,$$

where $\lfloor \cdot \rfloor$ denotes the integer-part function.

Certainly, (v_n) is bounded with $|v_n| \leq \operatorname{cosec}\theta$.

If $v_m = 0$ then $\theta = r\pi/m$ for some r, $1 \leq r \leq m-1$, and

$$v_{m+1} = \frac{\sin((m+1)r\pi/m)}{\sin(r\pi/m)} = (-1)^r.$$

Thus, if r is even then (v_n) is periodic (with period dividing m), and if r is odd then $v_{2m} = 0$ and $v_{2m+1} = 1$, so that (v_n) has period $2m$ (cf table 3 of Griffiths' article).

Thought of as a polynomial in k, $v_m(k) = 0$ has $m-1$ distinct nonzero roots given by $k = 2\cos(r\pi/m)$, $1 \leq r \leq m-1$. (Since, if m divides m', then the roots of $v_m(k) = 0$ appear among those of $v_{m'}(k) = 0$, and we confirm Griffiths' observation that, as polynomials in k, v_m divides $v_{m'}$, whenever m divides m'.)

The periodic sequences arise from $k = 2\cos(2\pi s/m)$, $1 \leq s < m/2$, and it is a standard result that, as illustrated in Griffiths' table 4, exactly $\frac{1}{2}\phi(m)$ of them – corresponding to s and m coprime – have period m, where $\phi(m)$ denotes Euler's totient. For example, if $m = 15$ and $\phi(15) = 8$ then the four entries (1.827, 1.338, –0.209, –1.956) in Griffiths' table 4 are revealed to be

$$2\cos\left(\frac{2\pi}{15}\right), \qquad 2\cos\left(\frac{4\pi}{15}\right), \qquad 2\cos\left(\frac{8\pi}{15}\right), \qquad 2\cos\left(\frac{14\pi}{15}\right).$$

Finally, it is worth recording that the bounded sequence

$$v_n = \frac{\sin n\theta}{\sin\theta}, \qquad 0 < \theta < \pi,$$

is never convergent. For the trigonometric identities

$$v_{n+1} - v_{n-1} = 2\cos n\theta,$$
$$\cos(n-1)\theta - \cos(n+1)\theta = 2\sin\theta\sin n\theta,$$

show that, if $v_n \to l$ then $\cos n\theta \to 0$ and $\sin n\theta \to 0$, contradicting the fact that $\sin^2 n\theta + \cos^2 n\theta = 1$.

Yours sincerely,

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