Dear Editor,

Sumlines of sides of regular n-gons

I enjoyed reading Jonny Griffiths' article *Sumlines* (*Math. Spectrum*, Volume 41, Number 2, pp. 50–56). His conjecture that the sumline of the sides of a regular n-gon (for odd n) always touches the incircle is quite correct.

As outlined in his article, we may suppose that the n-gon is centred at the origin of co-ordinates and inscribed in a circle of radius 1 with vertices

$$\left(\cos\left(\theta + \frac{2k\pi}{n}\right), \sin\left(\theta + \frac{2k\pi}{n}\right)\right), \qquad 0 \le k \le n-1.$$

Jonny's algebra for equilateral triangles on p. 55 readily extends to show that the sumline of the sides has the equation

$$ny = -\left(\sum_{k=0}^{n-1} \cot\left(\theta + \frac{\pi}{n}(2k+1)\right)\right)x + \cos\frac{\pi}{n}\left(\sum_{k=0}^{n-1} \operatorname{cosec}\left(\theta + \frac{\pi}{n}(2k+1)\right)\right)$$

or $ny = -Ax + \cos(\pi/n)B$, say.

This sumline is a perpendicular distance $|\cos(\pi/n)B|/\sqrt{n^2 + A^2}$ from the origin and, thus, touches the incircle (of radius $\cos(\pi/n)$) if

$$\frac{B}{\sqrt{n^2 + A^2}} = 1 \quad \text{or} \quad (B - A)(B + A) = n^2.$$

Since

$$\operatorname{cosec}\phi - \cot\phi = \frac{1-\cos\phi}{\sin\phi} = \tan\frac{\phi}{2} \quad \text{and} \quad \operatorname{cosec}\phi + \cot\phi = \frac{1+\cos\phi}{\sin\phi} = \cot\frac{\phi}{2},$$

we see that

$$(B-A)(B+A) = \left(\sum_{k=0}^{n-1} \tan\left(\frac{\theta}{2} + \frac{\pi}{2n} + \frac{k\pi}{n}\right)\right)\left(\sum_{k=0}^{n-1} \cot\left(\frac{\theta}{2} + \frac{\pi}{2n} + \frac{k\pi}{n}\right)\right).$$

Consider the equation $\tan nx - \tan(n\theta/2 + \pi/2) = 0$ with roots $x = \theta/2 + \pi/2n + k\pi/n$ (where k is any integer).

When n is odd, the expansion of $\tan nx$ in terms of $t = \tan x$ (given, for example, by equating real and imaginary parts of $\cos nx + \mathrm{i}\sin nx = (\cos x + \mathrm{i}\sin x)^n$) takes the form

$$\tan nx = \frac{\binom{n}{1}t - \binom{n}{3}t^3 + \dots + (-1)^{(n-1)/2}t^n}{1 - \binom{n}{2}t^2 + \dots + (-1)^{(n-1)/2}\binom{n}{n-1}t^{n-1}}.$$

Equating this to $\tan(n\theta/2 + \pi/2) = -\cot(n\theta/2)$ shows that $t = \tan(\theta/2 + \pi/2n + k\pi/n)$ for $k = 0, 1, \dots, n-1$ give the n roots of

$$(-1)^{(n-1)/2}t^n + (-1)^{(n-1)/2}\binom{n}{n-1}\cot\frac{n\theta}{2}t^{n-1} + \dots + \binom{n}{1}t + \cot\frac{n\theta}{2} = 0.$$

By standard relations between the roots and coefficients, we then see that

$$(B-A)(B+A) = \text{"}\sum \text{roots} \sum \frac{1}{\text{roots}}\text{"} = -n\cot\frac{n\theta}{2}\frac{-n}{\cot(n\theta/2)} = n^2,$$

as required.

Finally, it is perhaps worth remarking that we can give an alternative proof that the sumline always goes through the origin when n is even. Here, we need to show that $B = 0$, which is immediate from pairing the terms of the sum:

$$\sum_{k=0}^{n-1} \operatorname{cosec}\left(\theta + \frac{\pi}{n}(2k+1)\right) = \sum_{k=0}^{n/2-1}\left(\operatorname{cosec}\left(\theta + \frac{\pi}{n}(2k+1)\right) + \operatorname{cosec}\left(\theta + \frac{\pi}{n}(2k+1+n)\right)\right) = 0,$$

since $\operatorname{cosec} x + \operatorname{cosec}(x+\pi) = 0$.

Yours sincerely,

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