The Not-On-The-Line Transformation

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'My credibility is not on the line.' President Barack Obama, 4-9-13

In two dimensions, let us define a line $L_0 = 0$ not through the origin, where $L_0 = ax + by + c, c \neq 0$. Now choose a point $R = (p, q)$ with $p \neq 0, q \neq 0$, that does NOT lie on $L_0 = 0$. We will also insist that $ap \neq bq$, for reasons that will become clear. We can define a transformation T as follows: suppose R DOES lie on each of the the three lines $a'x + by + c = 0, ax + b'y + c = 0$ and $ax + by + c' = 0$. Now define the line $L_1 = 0$ as $a'x + b'y + c' = 0$. We will say that the transformation T, mapping lines to lines in the $x - y$ plane, is defined by $T(L_0) = L_1$, and we can define $L_n = T^n(L_0)$. It is also useful to define $s = ap + bq + r$, where we know $s \neq 0$.

Theorem 1. *The point R does NOT lie on $L_1 = 0$ either.*

Proof. We have $a'p + bq + c = 0, ap + b'q + c = 0$, and $ap + bq + c' = 0$. Adding these gives $a'p + b'q + c' + 2s = 0$. Since $s \neq 0$, we must have $a'p + b'q + c' \neq 0$. □

What is $L_1 = 0$ explicitly? We have $a'p + bq + c = 0$, and so $a' = \frac{-bq-c}{p}$. Similarly $b' = \frac{-ap-c}{q}$, and $c' = -ap - bq$. Thus L_1 is $\frac{-bq-c}{p}x + \frac{-ap-c}{q}y - ap - bq$, which is more conveniently written as

$$\frac{bq + c}{p}x + \frac{ap + c}{q}y + ap + bq = 0, \tag{1}$$

or $\frac{ap+bq+c}{p}x + \frac{ap+bq+c}{q}y + ap + bq + c - (ax + by + c) = 0$, or $s(\frac{x}{p} + \frac{y}{q} + 1) - L_0 = 0$. Let us put $\frac{x}{p} + \frac{y}{q} + 1 = M$, so $L_0 + L_1 = sM$. We note that the line $M = 0$ is independent of a, b, and c.

Given the lines $C_0 = 0$ and $C_1 = 0$, then $\alpha C_0 + \beta C_1 = 0$ either passes through the intersection point of $C_0 = 0$ and $C_1 = 0$, or is parallel to them both, so L_1 goes through the intersection of $L_0 = 0$ and $M = 0$, which we shall call Z.

The line $L_0 = 0$ could be written as $\lambda ax + \lambda by + \lambda c = 0$ for any non-zero real number λ. In this case, our image line $L_1 = 0$ is $\frac{\lambda bq+\lambda c}{p}x + \frac{\lambda ap+\lambda c}{q}y + \lambda ap + \lambda bq = 0$, which on dividing by λ is seen to be our original $L_1 = 0$, and so T is well-defined.

Note that by multiplying (1) by pq, we could say that T maps $ax+by+c=0$ to the line $q(bq+c)x+p(ap+c)y+ap^2q+bpq^2=0$. We could write this as

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 0 & p^2 & p^2q \\ q^2 & 0 & pq^2 \\ q & p & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0,$$

where the central matrix here has the eigenvalues $2pq$ and $-pq$ (twice).

What is L_2? This is $\frac{b'q+c'}{p}x + \frac{a'p+c'}{q}y + a'p + b'q$, which becomes $(a + \frac{ap+bq+c}{p})x + (b + \frac{ap+bq+c}{q})y + c + (ap+bq+c)$, and so $L_2 = L_0 + sM$. Given that $sM = L_0 + L_1$, we have $L_2 = L_1 + 2L_0$, which generalises to

$$L_{n+2} = L_{n+1} + 2L_n, n \geqslant 0. \tag{2}$$

Theorem 2. *The lines $L_0 = 0, L_1 = 0, L_2 = 0, L_3 = 0, \ldots$ are concurrent.*

Proof. From (2), each line L_n passes through the intersection of the previous pair, and so all are concurrent or parallel. This point is Z: solving $L_0 = 0$ and $M = 0$ simultaneously gives $Z = \left(\frac{p(bq-c)}{ap-bq}, \frac{q(c-ap)}{ap-bq}\right)$. This is the reason for excluding the case $ap = bq$; here $L_0 = 0$ and $L_1 = 0$, and indeed $L_n = 0$ for all n, are parallel to $M = 0$. □

Theorem 3. *The transformation T maps $M = 0$ to itself, and $M = 0$ is the only line for which this is true.*

Proof. We have $L_0 + L_1 = sM$, and so if $L_0 = k_0M$ for some constant k_0, then $L_1 = k_1M$. Also if $L_1 = kL_0$, then $(k+1)L_0 = sM$, and $L_0 = 0$ is the line $M = 0$. □

Can we find L_n explicitly? Let us define j_n to be the n^{th} Jacobsthal number, as given by $j_n = j_{n-1} + 2j_{n-2}, j_0 = 0, j_1 = 1$, (see OEIS A001045 starting 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341,...).

Theorem 4. *The line L_n is $j_nsM + (-1)^nL_0 = 0$.*

Proof. Our assertion is true for $n = 0$. Suppose it is true for some $n \geqslant 0$. The line L_n can therefore be written as

$$\left(\frac{j_ns}{p} + (-1)^na\right)x + \left(\frac{j_ns}{q} + (-1)^nb\right)y + j_ns + (-1)^nc = 0,$$

Using (1), the line L_{n+1} is therefore

$$\frac{2j_n s + (-1)^n bq + (-1)^n c}{p} x + \frac{2j_n s + (-1)^n ap + (-1)^n c}{q} y$$
$$+2j_n s + (-1)^n ap + (-1)^n bq = 0,$$

which is

$$\frac{(2j_n + (-1)^n)s - (-1)^n ap}{p} x + \frac{(2j_n + (-1)^n)s - (-1)^n bq}{q} y$$
$$+(2j_n + (-1)^n)s - (-1)^n c = 0.$$

We note that the number j_n is given by $\frac{2^n - (-1)^n}{3}$, and so $2j_n + (-1)^n = j_{n+1}$. So our equation becomes

$$j_{n+1} s \left(\frac{x}{p} + \frac{y}{q} + 1 \right) + (-1)^{n+1}(ax + by + c) = 0,$$

and so by induction the proof is complete. We see clearly again that L_n will always go through the intersection point Z of $L_0 = 0$ and $M = 0$. □

As an alternative proof, we can examine the recurrence relation (2).

Proof. The auxiliary equation is $\lambda^2 - \lambda - 2 = 0$, so $\lambda = 2$ or -1, and $L_n = A(2)^n + B(-1)^n$. Now from (2),

$$L_{n+2} + L_{n+1} = 2(L_{n+1} + L_n) \ldots = 2^{n+1}(L_1 + L_0) = 2^{n+1} sM.$$

Now $L_{n+2} + L_{n+1} = A2^{n+2} + B(-1)^{n+2} + A2^{n+1} + B(-1)^{n+1} = 2^{n+1}(3A)$. Thus $2^{n+1}(3A) = 2^{n+1} sM$, and $A = \frac{sM}{3}$. Now $L_1 = 2A - B = sM - L_0$, and so $B = L_0 - \frac{sM}{3}$. So we have

$$L_n = (-1)^n(ax + by + c) + \frac{2^n - (-1)^n}{3} sM = (-1)^n(ax + by + c) + j_n sM,$$

as before. □

The gradient of $L_n = 0$ is $(-\frac{q}{p})\frac{j_n s + (-1)^n ap}{j_n s + (-1)^n bq}$. Since j_n tends swiftly to infinity, this tends swiftly to $-\frac{q}{p}$, or the gradient of M. So we now have a clear picture of what happens as we apply the transformation T repeatedly to a line $L_0 = 0$. The point R is NOT on the line $M = 0$, but the point $(-\frac{p}{2}, -\frac{q}{2})$

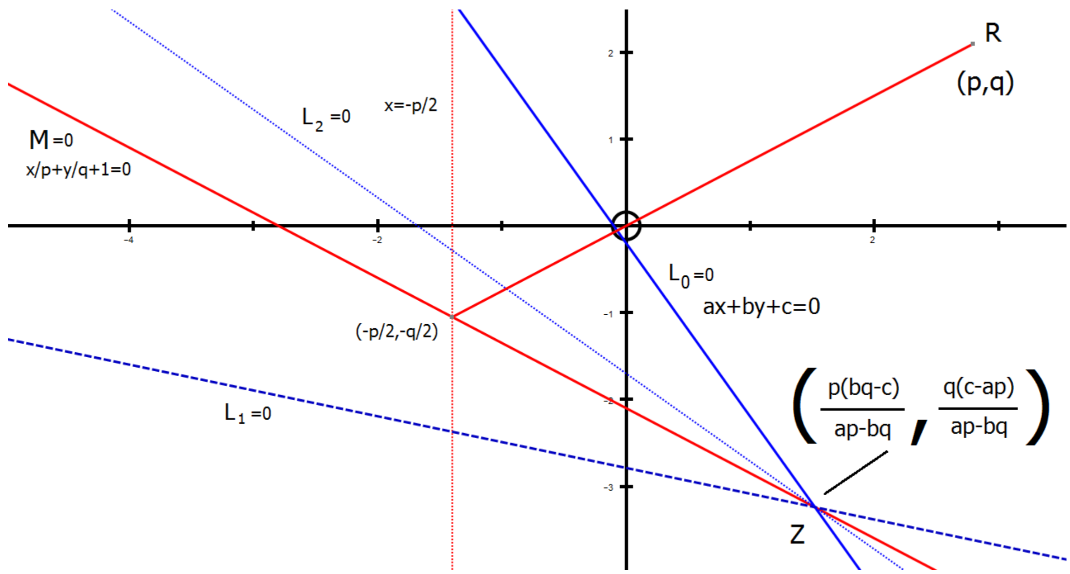

Figure 1: The transformation T

is. The gradient of the line joining these two points is $\frac{q}{p}$, while the gradient of M is $-\frac{q}{p}$. Thus we have the red lines in Figure 1, with $x = -\frac{p}{2}$ as a line of symmetry.

The line $L_0 = 0$ crosses the line $M = 0$ at Z, as do all the image lines $L_n = 0$. The lines $L_n = 0$ converge on the line $M = 0$ as n increases, oscillating about it as they do. The exceptional case is when $L_0 = 0$ and $M = 0$ are parallel, but here too the image lines $L_n = 0$ (all parallel to $M = 0$) converge on $M = 0$.

Unconventionally, we began here with a transformation T mapping lines to lines (the more usual approach would be to define how a transformation mapped points, and move on to lines from there). If intersection points map to intersection points for our transformation T, then starting with the non-parallel lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, we find these meet at $(X, Y) = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$. The two lines map to

$$(b_1q + c_1)\frac{x}{p} + (a_1p + c_1)\frac{y}{q} + a_1p + b_1q = 0$$

and

$$(b_2q + c_2)\frac{x}{p} + (a_2p + c_2)\frac{y}{q} + a_2p + b_2q = 0,$$

and these meet at (X', Y'). It is now possible to eliminate a_i, b_i and c_i to give

$$(X', Y') = \left(\frac{-pqX + p^2Y + p^2q}{qX + pY - pq}, \frac{q^2X - pqY + pq^2}{qX + pY - pq}\right).$$

This is a projective transformation, since it is of the form

$$(X', Y') = \left(\frac{\alpha X + \beta Y + \gamma}{\delta X + \epsilon Y + \phi}, \frac{\theta X + \chi Y + \nu}{\delta X + \epsilon Y + \phi}\right).$$

We can note that the point (p, q) is invariant under this transformation, and so is any point on the line $M = 0$. Fixing just one more point fixes the whole plane.

There are many ways that this investigation could be extended – we could, for example, work in three dimensions. Take a point (p, q, r) that is NOT on the plane $ax + by + cz + d = 0$, and so on; in this case the central matrix that emerges has eigenvalues $3pqr$ and $-pqr$ (three times). A simple question accessible to A Level students is this one:

1. Suppose the point (p, q) is INSIDE the circle $(x - a)^2 + (y - b)^2 = r^2$.
2. The point (p, q) is ON the three circles $(x - a')^2 + (y - b)^2 = r^2$, $(x - a)^2 + (y - b')^2 = r^2$ and $(x - a)^2 + (y - b)^2 = r'^2$, where $a' < a$, $b' < b, r' < r$.
3. Show (p, q) is OUTSIDE the circle $(x - a')^2 + (y - b')^2 = r'^2$.

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