Pearl Tilings

The other day I found myself considering the following tessellation:

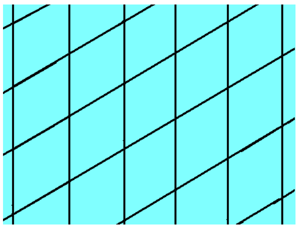


Fig. 1

I wondered: what happens if I throw a single regular hexagon into its midst? There is quite a disturbance, but the original tiles do manage to rearrange themselves around the new tile to create the following:

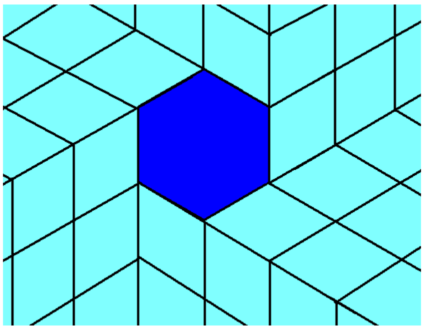


Fig. 2

I decided to call the tessellation in Figure 2 'a pearl tiling'. The shapes in Figure 1 I called 'the oyster tiles' (initially in their 'oyster tiling'), while the single added tile I called 'the iritile'. The reasons for my naming are I hope self-evident: the oyster tiles respond to the introduction of the iritile (in nature, a piece of shell, coral or bone - pearls are hardly ever formed around sand) by surrounding it, creating a fresh pattern. The result can, like a pearl, be beautiful. 'Iri-' is short for 'iridescent' as well as helpfully suggesting 'irritation'.

Questions immediately arise. Can any n-sided regular polygon be a successful iritile? What are the most efficacious shapes for oyster tiles? Can the same oyster tiles surround several different iritiles?

The rhombus in Figure 1 has angles of 60^o and 120^o. What if we generalise this to the rhombus with two angles of a^o and two of $(180 – a)^o$? If we have a regular n-agon, then if $a = \frac{360}{n}$, the n-agon will be an iritile. But if we fit several (say p) rhombuses into each external angle (as in Figure 3 where p = 3) then $a = \frac{360}{np}$ and we have a family of rhombuses that will each create a pearl tiling for this polygon.

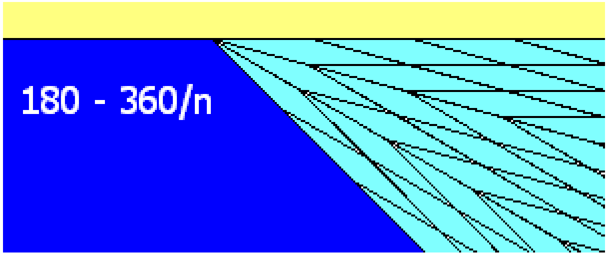


Fig. 3

It is easy to see that such a rhombus can be an oyster tile for several regular n-agons, indeed as many different regular n-agons as we like. For example, suppose we require an oyster tile that will surround a 7-agon, an 11-agon, and a 13-agon. Pick a = $\frac{360}{7 \times 11 \times 13}$. Then with p = 11×13, the 7-agon is surrounded, with p = 7×13 the 11-agon is surrounded, and with p = 7×11, the 13-agon is surrounded. If we choose a = $\frac{1}{7}^{o}$, the resulting oyster tile will surround a regular 3, 4,… through to a 10-agon.

Isosceles triangles form natural oyster tiles too. Figure shows the formation of a pearl tiling with five-fold symmetry for a regular pentagon.

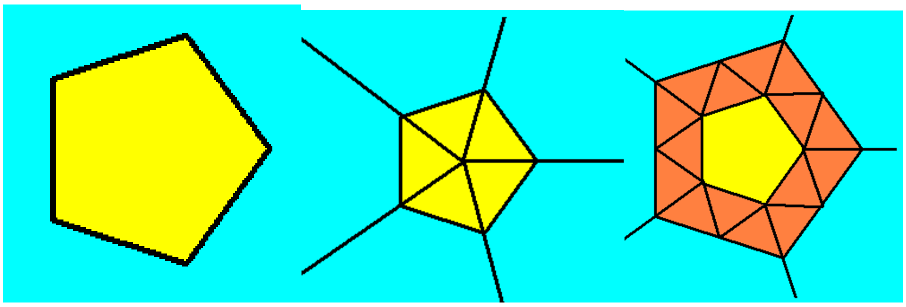


Fig. 4

The triangles in Figure 4 can tile to infinity, and it is clear that any regular polygon can be a successful iritile in this way. But this is not the only possible isosceles triangle that will work. Just as there are a family of rhombuses that will act as oyster tiles for any regular polygon, so there are a family of isosceles triangles that will do the same.

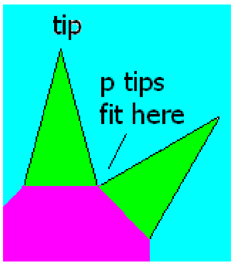


Fig. 5

Given a regular n-agon, we need 180 – 360/n + 2a + p(180 - 2a) = 360.

So a = $90 - \frac{180}{n(p-1)}$. Any isosceles triangle with a base angle of this type will tile the rest of the plane, since 4a + 2(180 - 2a) = 360 (Figure 6) whatever the value of a may be.

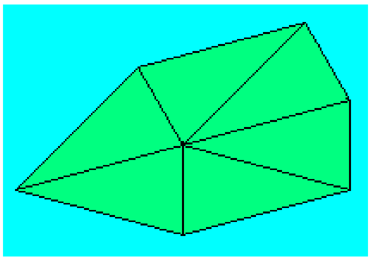


Fig. 6

Shapes of the following hexagonal design also prove to be willing oyster tiles, their success due to the fact that 2b + a = 360.

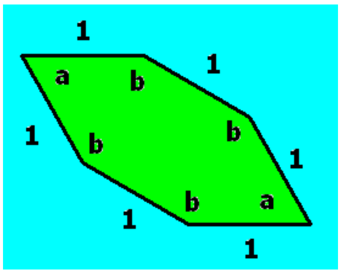


Fig. 7

Figure 8 gives an example of one of these oyster tiles in action.

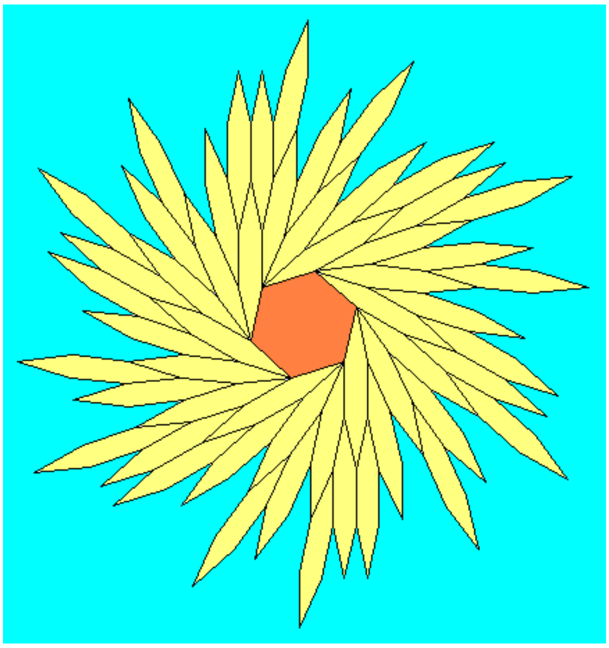
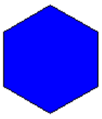
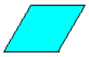
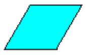
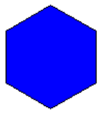


Fig. 8

It might be helpful to introduce some notation at this point. We can define the binary relation S_1 **.o** S_2 to denote the fact that S_1 is an iritile for the oyster tile S_2. We might first note that given any tile T that tessellates, T **.o** T, trivially, so our new relation is reflexive. Is it transitive? That is if S_1 **.o** S_2 and S_2 **.o** S_3, does S_1 **.o** S_3? Initial research suggests the answer might be yes: this is certainly a question to investigate further.

If S_1 **.o** S_2, does S_2 **.o** S_1? Not necessarily. From Figure 1,

 .O  , but we do not have  .O  .

Is it possible for S_1 **.o** S_2 and S_2 **.o** S_1 to be true together non-trivially? Yes:

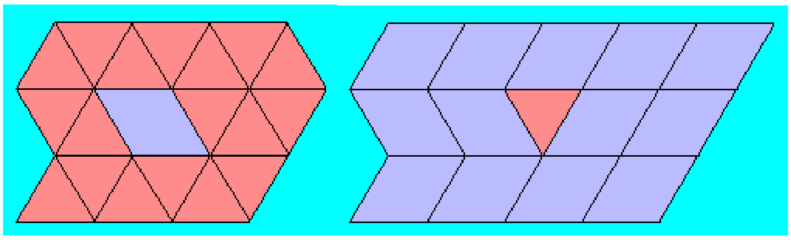


Fig. 9

If S_1 **.o** S_2 and S_2 **.o** S_1, we can define a new relation, S_1 **.o.** S_2.

Considering polyominoes at this point is a good idea. There are only two triominoes, let us call them Tm_1 and Tm_2, and it is easy to see that Tm_1 **.o.** Tm_2.

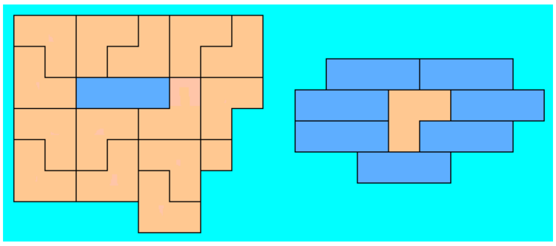


Fig. 10

What of the five quadrominoes? Pick any two, and you will find that the relation **.o.** holds.

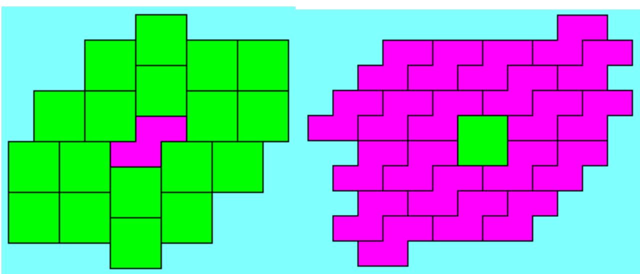


Fig. 11

Imagine yourself placed somewhere on one of these tilings at random. Unless you were fairly central, it would be hard to discern the dislocation caused by the iritile. Exploring the tiles in your immediate vicinity, you would probably conclude that the tiling was uniform. In some sense, the iritile 'never happened'.

There are twelve pentominoes, and the relation **.o.** certainly sometimes holds.

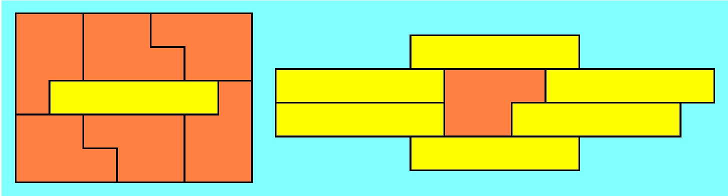


Fig. 12

But will it always do so? Another fruitful question to investigate.

One last question: is there a pair of distinct triangles T_1 and T_2 so that T_1 **.o.** T_2? The best candidates would seem to be an isosceles pair, with sides as in Figure.

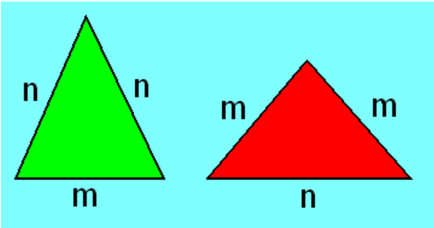


Fig. 13

The most famous triangles of this type are the following:

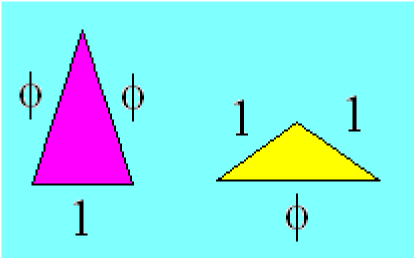


Fig. 14

And indeed, when you experiment, you find that T_1 **.o.** T_2 does hold for this pair.

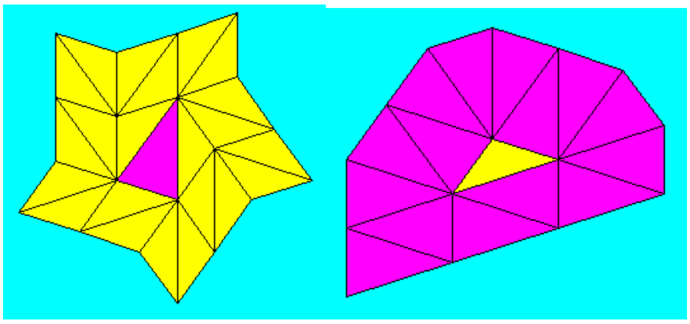


Fig. 15

One more reason to celebrate the Golden Ratio!

April 2006, Jonny Griffiths, Paston College, jonny.griffiths@paston.ac.uk