Could π be three?

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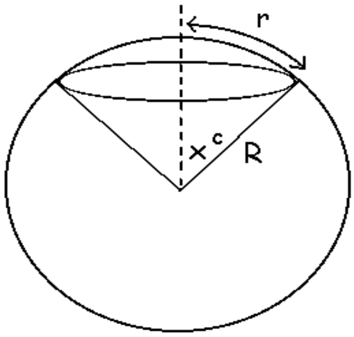
How does my title arise? As many readers will know, there are one or two verses in the Bible that seem to indicate that the value of π, understood as the circumference of a circle divided by its diameter, is 3 exactly. (For an illuminating discussion on this, see Deakin and Lausch [1].)

"And he made a molten sea, ten cubits from the one brim to the other: it was round all about, […] and a line of thirty cubits did compass it round about." [2]

This describes King Solomon building the Temple in Jerusalem. In playful moments, I have tried to imagine a world in which these words might be literally true, and not just an approximation to the truth. Maybe Solomon, who was alive long before Euclid, was using a different metric to the Euclidean one, and maybe with a little effort his alternative geometry might be reconstructed. Let us start by giving our usual Euclidean π the name π_E , and rename π appropriately in other geometries as we go.

What would be the easiest alternative geometry that we could begin with? We might turn to spherical geometry, where points are points on the surface of a sphere, and distance is measured on the surface of the sphere. A circle in this geometry is the set of points on the surface of the sphere equidistant from one point on the surface: in other words, a circle in this geometry looks just like a circle in Euclidean geometry. So what will π_S be? Take a circle of radius r (measured in this new geometry) subtending an angle $2x^c$ at the centre of the sphere.



Then r = Rx, and the circumference of the circle is $2\pi_E R \sin x$, which gives:

$$\pi_S = \frac{2\pi_E R \sin x}{2Rx} = \frac{\pi_E \sin x}{x}$$

So π_S varies with x. When x = 0, then $\pi_S = \pi_E$, and when $x = \frac{\pi_E}{2}$, $\pi_S = 2$. π_S can take any value between these extremes. Of especial interest to us is the fact that when $x = \frac{\pi_E}{6}$, then $\pi_S = 3$. So we have our Euclidean geometry where π is constant, and now another where π is 3 at least some of the time. In the light of this, a geometry where π is constant at 3 seems more possible, perhaps.

Euclidean geometry arises from saying that there is just one line through a point parallel to a line not through that point. Spherical geometry says there are no lines through a point parallel to a given line not through the point. So what of hyperbolic geometry (sometimes called Non-Euclidean geometry, a little unhelpfully, since there are many Non-Euclidean geometries), the geometry that arises from saying that there are infinitely many lines parallel to a line through a given point not on that line? What is π there? Fortunately there is a marvellous website that allows us to explore hyperbolic geometry, based at The University of New Mexico and run by Joel Castellanos. (The address is: *http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html)* Here we can draw hyperbolic circles, and see roughly what their circumferences are, and measure their diameters. It is easy to see from experimenting that for this geometry, $\pi_E < \pi_H < \infty$, apparently little help in our search for a system where π is three.

The idea of distance used in this last geometric world is quite complicated. To be exact, if two points (seen as complex numbers inside the unit circle) are z_1 and z_2, then:

$$d(z_1, z_2) = \tanh^{-1}\left(\left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right| \right)$$

What makes this satisfactory as a distance measure? What should we insist upon in our definition of distance, and are there any restrictions that we should impose? The classic definition of a distance function (or metric) d is that:

$d(z_1, z_2) \geq 0$ (**any distance is non-negative)**
$d(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$ **(the distance from A to B is zero iff A = B)**
$d(z_1, z_2) = d(z_2, z_1)$ **(the distance from A to B is the distance from B to A)**
$d(z_1, z_2) + d(z_2, z_3) \geq d(z_1, z_3)$ **(the distance from A to C via B ≥ the direct distance from A to C)**

These all seem to be reasonable things to request. So what other ideas of distance, apart from the three that we have met so far, fit these criteria? There are two in particular that might be useful to us.

1. The Manhattan metric (here called d_M)…imagine you are in New York, and you can only travel from A to B by moving along gridlines. Cutting corners is not allowed. More formally, the distance from (a , b) to (c , d) is $|a-c| + |b-d|$, where a, b, c and d can take any real values.

2. The Chessboard metric (here called d_C)…imagine you are on a large chessboard, and two rooks are placed upon it. The distance from one rook to the other is the length of the single move required to bring them as close together as possible. More formally, the distance from (a , b) to (c , d) is the larger of $|a - c|$ and $|b - d|$, where again, a, b, c and d can take any real values.

So what might π_M and π_C be? We can draw a unit circle, centre the origin, for each of these metrics. The Manhattan metric leads to a 'square' (figure 1), with corners at (1,0), (0,1), (-1,0), and (0,-1).

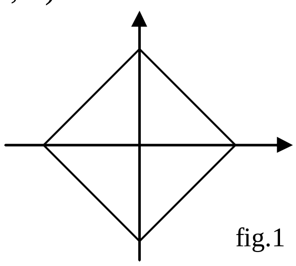


fig.1

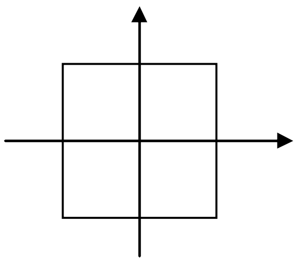


fig.2

What is the circumference C of this 'circle'? (Incidentally, try telling students that this is a circle!) Continuing to use this metric, clearly C = 8, whilst the diameter is 2. (A diameter is defined as a straight line connecting two points on the circle that is maximal in length. In Euclidean geometry, there is a unique straight line between two points, but not so here: there can be many diameters connecting two suitable points on the circle, but they will all be of length 2.) Thus $\pi_M = 4$, and it is not too hard to see that this will be the value of π_M for any circle in this geometry.

The Chessboard metric also gives a 'square' (figure 2) as the unit circle, centre the origin, but the corners here are (1,1), (-1,1), (-1,-1) and (1,-1). Once again, using this metric gives a side length of 2, a circumference of 8, and a diameter of 2. (Again, there are many diameters connecting two suitable points on opposite sides of the circle, all of length 2.) So $\pi_C = 4$, for all circles in the system. If Solomon had given ten and forty as his measurements, we would not know which geometry he might have been using.

If d_A and d_B are metrics, then $ad_A + bd_B$ will be a metric too. (This is easy to verify from our definition of a metric.) So what happens if we take say, $ad_M + (1–a)d_C$ as our metric, with $0 < a < 1$? (Call this d_{MCa}.) We might think that since both π_M and π_C are 4, then π_{MCa} must equal 4 too, yet not so. What does the unit circle centre the origin look like with this new metric? (Choosing (1–a) as the second coefficient ensures that the circle goes through (1,0)).

Take a point (x , y) on the circle in the first quadrant with x > y.

$$1 = a(x + y) + (1-a)\,x \quad \Rightarrow y = \frac{(1-x)}{a} \Rightarrow \text{when } x = y,\ x = y = \left(\frac{1}{1+a}\right)$$

$\Rightarrow$ the straight line from (1,0) to $\left(\frac{1}{1+a}, \frac{1}{1+a}\right)$ is part of the circle. Thus the circle is an octagon (not a regular one), with each side length $a\left(\frac{1}{1+a} + 1 - \frac{1}{1+a}\right) + (1-a)\left(\frac{1}{1+a}\right)$, that is, $\frac{a^2+1}{a+1}$, which gives a circumference of $8\left(\frac{a^2+1}{a+1}\right)$. The diameter is still always 2, so $\pi_{MCa} = 4\left(\frac{a^2+1}{a+1}\right)$.

Putting this equal to 3 gives $4a^2 – 3a + 1 = 0$, which has imaginary roots: still no joy! So what is the smallest that π_{MCa} can be here? It turns out that the minimum occurs when a = 0.417… giving π_{MCa} = 3.31…, closer to 3 than π_M or π_C, but still not close enough.

However, we’ve not come out of this with nothing. For if we could find a metric where π is constant and less than three, it seems we might be able to find a linear combination of that metric with another to give one where π is always three.

Maybe to get π to be always 3 we need to think differently. Consider the following finite geometry:

Take the four numbers, 5, 6, 12, and 15 as our points. In binary these are 0101, 0110, 1100, and 1111. So if the distance apart of two points is the number of digits where they disagree, then any of these numbers is 2 away from any other. Picking any of these numbers as a centre, and choosing radius 2, we find the other three numbers to be on the resulting 'circle'. If 'circumference' is defined as the length of the minimal loop that connects the points on the 'circle', then I have a circumference of 2 + 2 + 2 = 6. 'Diameter' still means the maximal distance between two points on the circle, and that is 2. $\pi = \frac{C}{D} = \frac{6}{2} = 3$. This holds for all four 'circles' in the system. Does our definition of distance here (called the Hamming distance, after its inventor) obey our rules for a metric? Happily, it does.

What if we take more than four points? If we have n points, then there will be $\frac{n(n-1)}{2}$ connecting lines, of various lengths. If we assume that these split into the kind of circles that we have above, each with six arcs, then $6 \mid \frac{n(n-1)}{2}$. But also, each point will be attached to three others for each of these circles. So $3 \mid (n–1)$. This gives 13 as the smallest value for n > 4 that might work, and we have the conjecture:

There is a geometry of thirteen numbers, that includes circles with thirteen different radii, such that every circle in the system has a value for π *of three.*

If this is true, the numbers will be huge!

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References

1. M.A.B Deakin and H. Lausch, the Bible and Pi, *The Mathematical Gazette* July 1998, Volume 82, Number 494.

2. I Kings, chapter 7 verse 3.