Pineapple Triangles

Jonny Griffiths documents a mathematical collaboration between himself and Derek Ball.

It all started, like so many beautiful things, over lunch at the Association of Teachers of Mathematics conference. Derek and I had been discussing the skills required to be a stand-up comic, when the conversation somehow turned back to mathematics. “Did you know,” Derek asked, “That if you take a triangle that includes the angles *a* and *2a*, the sides obey a law close to Pythagoras?” This was news to me, and I filed it away under ‘Investigate later,’ a file that is always bulging as conference draws to a close. Once I’d trekked back home, I tried for myself.

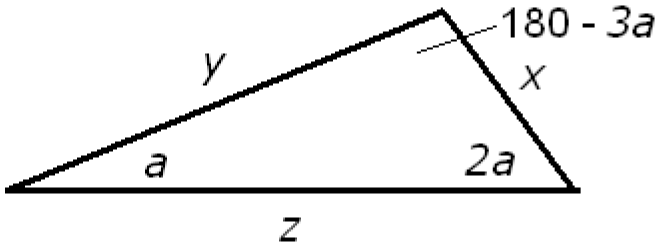


Fig. 1

I could see that by the Sin Rule, the sides (x, y, z) would be $\left(x, x\frac{\sin(2a)}{\sin(a)}, x\frac{\sin(3a)}{\sin a}\right)$ and thus in the ratio

sin a : sin $2a$: sin $3a$, but how could this be related to anything like Pythagoras? I emailed Derek urgently. “Try similar triangles; trig is not required!” was all the hint I was offered. It was, fortunately, all I needed.

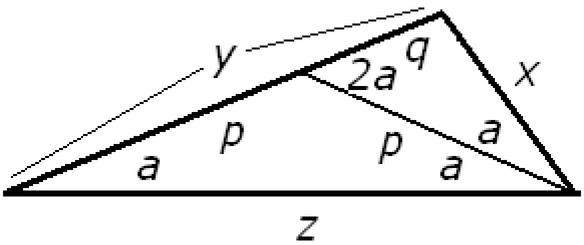


Fig. 2

The triangles with sides (x, y, z) and (q, x, p) are similar here, which gives:

$$\frac{q}{x} = \frac{x}{y} = \frac{p}{z}\text{, which means that } y = p + q = \frac{xz}{y} + \frac{x^2}{y}\text{, and so } \boldsymbol{y^2 = x^2 + xz} \ (1)$$

Close to Pythagoras indeed! But I was left wondering, what next? Derek told me there were plenty of integer values that fitted equation (1); would the same be true for a triangle containing angles *a*, and *3a*?

a and *4a*? How about an *a, na* triangle?

‘This looks worryingly close to “banana”,’ I said to Derek.

‘It’s nearer to “ananas” – let’s call these “pineapple triangles”,’ he replied. So we did.

Now I confess, I have no guilt over turning to a computer here - Excel is an awesome program, that devours this type of search problem. But I needed a way to connect together the sides for an *a*, *na* triangle.

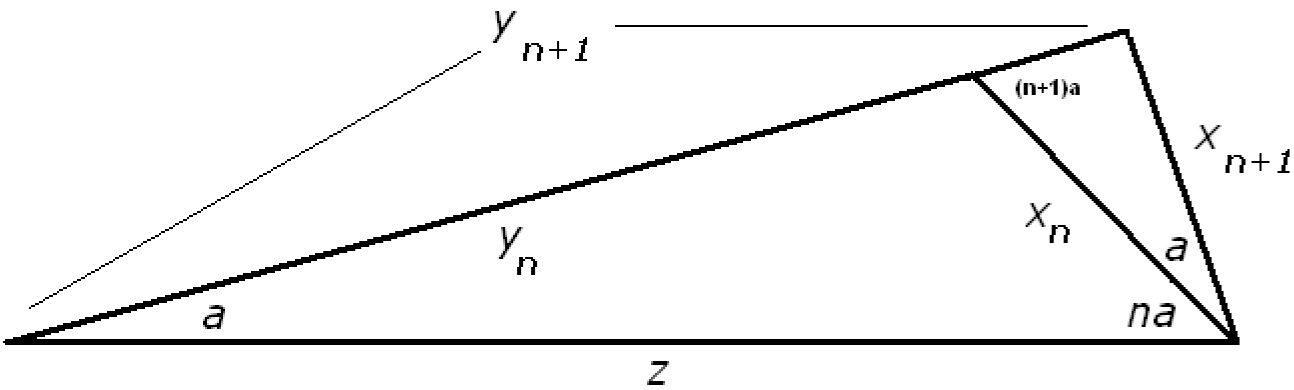


Fig. 3

Again we have a pair of similar triangles here, which give;

$$\frac{y_{n+1} - y_n}{x_{n+1}} = \frac{x_{n+1}}{y_{n+1}} = \frac{x_n}{z} \text{, which means that } x_n = \frac{z x_{n+1}}{y_{n+1}}, y_n = \frac{y_{n+1}{}^2 - x_{n+1}{}^2}{y_{n+1}} \quad (2)$$

So $y^2 = x^2 + xz$ turns into $y_2{}^2 = x_2{}^2 + x_2 z$, and substituting in we get

$$\left(\frac{y_3{}^2 - x_3{}^2}{y_3{}^2}\right)^2 = \frac{z^2 x_3{}^2}{y_3{}^2} + \frac{z^2 x_3}{y_3} \quad (3)$$

Repeating the process gives an even nastier connection between x_4, y_4 and z – not nasty enough to defeat Excel completely, however, which yielded results with a search program that included these;

(a, 2a) pineapple triangles			(a, 3a) triangles		
x	y	z	x	y	z
4	6	5	8	10	3
9	12	7	27	48	35
9	15	16	64	132	119
16	20	9	125	195	112
16	28	33	184	230	69
25	30	11	125	280	279
25	35	24	232	290	87
25	40	39	248	310	93
36	42	13	343	357	20
25	45	56	296	370	111
49	56	15	328	410	123
49	63	32	344	430	129
36	66	85	(a, 4a) triangles		
49	70	51	x	y	z
64	72	17	81	105	31
64	104	105	256	476	305

Table 1

I was trying to collect here examples that I called ‘primitive’, where *x, y* and *z* had no common factor. (Clearly a triangle with sides *x, y* and *z* is an (*a*, *na*) triangle, if and only if a triangle with sides k*x*, k*y*, and kz is one.) Certainly there were plenty of primitive (*a*, *2a*) pineapple triangles, but it got harder to find primitive (*a, na*) examples the larger n became. Another curiosity; for (*a*, *2a*) triangles, *x* seemed to be a square almost all of the time, while for (*a*, *3a*) triangles, *x* was more often than not a perfect cube, and so on. Sadly, the pattern did not seem to work consistently; but then Derek pointed out, ‘Not all your examples are primitive here!’ There was a glitch in my programming that had allowed some triangles through with

$\gcd(x, y, z) \neq 1$. Once these were removed, the square, cube, and fourth power patterns for side *x* were perfect. I reflected afterwards that given a choice between considering that I might have made an error in my program, and ditching a lovely conjecture, I had chosen the latter…

Now suddenly there was a focus to our work. Our possible theorem became this;

If a triangle with integer sides (*x, y, z*) (where *x, y,* and *z* have no common factor) contains the angles *a* and *na* (where *a* is a real number and opposite the side *x*, and where *n* is a positive integer), then *x* will be of the form k^n, where *k* is an integer.

Derek had worked on parametrising our starting equation (1), discovering that $(x, y, z) = (a^2, ad, d^2 - a^2)$ where *a* and *d* were positive integers with $\gcd(a, d) = 1$ gave integer solutions. I pointed out that we needed $a < d < 2a$ for this to give three legitimate sides of a triangle, and I had an additional niggle to resolve. “Your parametrisation always gives a solution to (1), but does it give them all?” I asked. The following elegant argument arrived from Derek;

Suppose that $y^2 = x^2 + xz$*, where* $\gcd(x, y, z) = 1$*. Let* $\gcd(x, y) = c$*, so* $x = ac$*,* $y = dc$*, with* $\gcd(a, d) = 1$*.*

$$z = \frac{y^2 - x^2}{x} = \frac{d^2c - a^2c}{a}$$ *which means* $a \mid d^2c$*, and thus* $a \mid c$*. Say* $c = ak$*.*

So $x = a^2k$*,* $y = adk$*, and* $z = (d^2 - a^2)k$*, and since k is a factor of x, y and z,* $k = 1$*.*

So a complete parametrisation of (1) is $(x, y, z) = (a^2, ad, d^2 - a^2)$*.*

How to find parametrisations for larger values of n? Derek had the great idea of reversing my recurrence relations (2), to give;

$$x_{n+1} = \frac{zx_ny_n}{z^2 - x_n{}^2}, y_{n+1} = \frac{z^2y_n}{z^2 - x_n{}^2} \quad (4)$$

So this gave a way to find – parametrisations for the higher situations. Put $n = 2$ and substitute $x_2 = a^2$, $y_2 = ad$, and $z_2 = d^2 - a^2$, and the above yields, after removing common factors and fractions;

$$x_3 = a^3,\ y_3 = a(d^2 - a^2),\ z_3 = d(d^2 - 2a^2)$$

So given that gcd(a, d) = 1, we have x, y and z are integers, gcd(x, y, z) = 1 and y is a perfect cube. Derek forged on, during an opportune holiday in the Lakes, to show that for n = 4, y was a perfect fourth power, with;

$$(x_4, y_4, z_4) = (a^4,\ d^3a - 2da^3,\ d^4 - 4d^2a^2 + 3a^4)$$

and that for $n = 5$, y was a perfect fifth power, with;

$$(x_5, y_5, z_5) = (a^5,\ d^4a - 3d^2a^3 + a^5, d^5 - 5d^3a^2 + 3da^4)$$

By now, the algebra was proving tiresome, and the baton passed to me, as someone lucky enough to possess a copy of the algebraic manipulation package *Derive*. As Derek paused to look after his grandchildren for a week, his final words were these;

"The polynomials for the parametrisations for x, y and z must fall into a pattern. If we can find this, I reckon we can prove by induction that it holds for all n."

My week ahead proved to be busy. I started with noticing that the recurrence relation (4) is perhaps best written (after removing common factors and fractions);

$$x_{n+1} = x_n y_n,\ y_{n+1} = y_n z_n,\ z_{n+1} = z_n^{\,2} - x_n^{\,2} \quad (5)$$

My conjecture was that our parametrisation would always look like the following:

$$x_n = a^n,\ y_n = az_{n-1},\ z_n = \mathrm{P}_n(d, a),$$

where $\mathrm{P}_n(d, a)$ is the polynomial sequence we had to determine. (Certainly pattern-spotting confirmed this for n = 1 to 9.) If this was correct, then the inductive step using (5) would become;

$$x_{n+1} = a^n(az_{n-1}),\ y_{n+1} = (az_{n-1})z_n,\ z_{n+1} = (\mathrm{P}_n(d, a))^2 - a^{2n} = z_{n-1}\,\mathrm{P}_{n+1}(d, a)$$

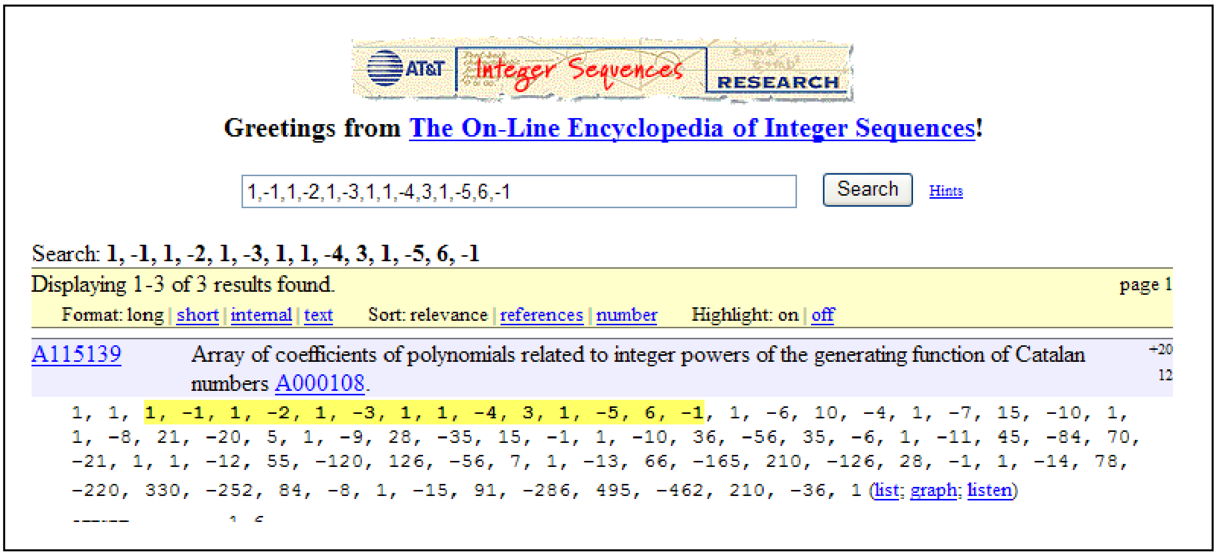
whereupon we could cancel z_{n-1} to give what we needed.

Derive made the first few polynomials $\mathrm{P}_n(d, 1)$ easy to find using (5) (I decided to leave out the 'a's, which simply made the polynomials homogeneous – it would be no problem to put them back);

2. $d^2 - 1$
3. $d^3 - 2d$
4. $d^4 - 3d^2 + 1$
5. $d^5 - 4d^3 + 3d$
6. $d^6 - 5d^4 + 6d^2 - 1$
7. $d^7 - 6d^5 + 10d^3 - 4d$
8. $d^8 - 7d^6 + 15d^4 - 10d^2 + 1$
9. $d^9 - 8d^7 + 21d^5 - 20d^3 + 5d$
10. $d^{10} - 9d^8 + 28d^6 - 35d^4 + 15d^2 - 1$

Table 2

I turned now to The Encyclopedia of Integer Sequences, the mathematician's friend, and certainly a candidate for the most useful mathematics website on the Net. Putting in the coefficients of our polynomials here gave the following;


Integer Sequences RESEARCH

Greetings from The On-Line Encyclopedia of Integer Sequences!

1,-1,1,-2,1,-3,1,1,-4,3,1,-5,6,-1 Search Hints

Search: **1, -1, 1, -2, 1, -3, 1, 1, -4, 3, 1, -5, 6, -1**

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A115139 Array of coefficients of polynomials related to integer powers of the generating function of Catalan numbers A000108. +20 12

1, 1, 1, -1, 1, -2, 1, -3, 1, 1, -4, 3, 1, -5, 6, -1, 1, -6, 10, -4, 1, -7, 15, -10, 1, 1, -8, 21, -20, 5, 1, -9, 28, -35, 15, -1, 1, -10, 36, -56, 35, -6, 1, -11, 45, -84, 70, -21, 1, 1, -12, 55, -120, 126, -56, 7, 1, -13, 66, -165, 210, -126, 28, -1, 1, -14, 78, -220, 330, -252, 84, -8, 1, -15, 91, -286, 495, -462, 210, -36, 1 (list; graph; listen)

Fig. 4

The site pointed me towards the Chebyshev Polynomials of the Second Kind, Un(x), which begin like this;

$$U_0(x) = 1$$

$$U_1(x) = 2x$$

$$U_2(x) = 4x^2 - 1$$

$$U_3(x) = 8x^3 - 4x$$

$$U_4(x) = 16x^4 - 12x^2 + 1$$

$$U_5(x) = 32x^5 - 32x^3 + 6x$$

$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$$

$$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$$

$$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$$

$$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$$

Table 3

Now we can see by inspection that the polynomials $P_n(d, 1)$ we have above in Table 2 are $U_n\left(\frac{d}{2}\right)$.

How is $U_n(x)$ defined? As follows:

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$$

In other words, expand sin((n+1)θ) into a polynomial in cos θ and sin θ, then divide by sin θ and then replace every cos θ by x - this gives you $U_n(x)$. Can we now carry out our induction? The inductive step asks us to prove;

$$P_{n+1}(d, a) = \frac{P_n(d,a)^2 - a^{2n}}{P_{n-1}(d,a)}$$

which in our new language becomes:

$$U_{n+1}\left(\frac{d}{2}\right) = \frac{\left(U_n\left(\frac{d}{2}\right)\right)^2 - 1}{U_{n-1}\left(\frac{d}{2}\right)}$$

So if we replace $\frac{d}{2}$ with cos(θ):

$$\frac{\left(U_n\left(\frac{d}{2}\right)\right)^2 - 1}{U_{n-1}\left(\frac{d}{2}\right)} = \frac{(U_n(\cos\theta))^2 - 1}{U_{n-1}(\cos\theta)} = \frac{\frac{(\sin((n+1)\theta))^2}{\sin^2\theta} - 1}{\frac{\sin(n\theta)}{\sin\theta}}$$

$$= \frac{\sin^2((n+1)\theta) - \sin^2\theta}{\sin\theta\sin(n\theta)} = \frac{(\sin((n+1)\theta) - \sin\theta)(\sin((n+1)\theta) + \sin\theta)}{\sin\theta\sin(n\theta)}$$

$$= \frac{2(\cos(\frac{n+2}{2}\theta)\sin\frac{n}{2}\theta)2(\sin(\frac{n+2}{2}\theta)\cos\frac{n}{2}\theta)}{\sin\theta\sin(n\theta)} = \frac{\sin n\theta\sin((n+2)\theta)}{\sin\theta\sin(n\theta)}$$

$$= \frac{\sin((n+2)\theta)}{\sin\theta} = U_{n+1}\left(\frac{d}{2}\right).$$

And so we have it. A proof of our conjecture; a little rough at the edges maybe, with polishing definitely required, but nonetheless, a proof. I preened a little as I posted this off to Derek, who was just about to emerge from his long shift as grandfather.

“Lovely, Jonny!” he said, but his next comment contained a friendly barb. “By the way,” he said innocently, “You do realise that the θ you talk about in your proof is the same as the ‘a’ in your starting triangle?”

I felt deflated, crestfallen, even shocked. I went all the way back to my initial thoughts – that the sides would be in the ratio sin a : sin $2a$: sin $3a$. Of course! From Fig. 1, $(x, y, z) = \left(x, x\frac{\sin(2a)}{\sin a}, x\frac{\sin(3a)}{\sin a}\right)$ that is,

$(x, xU_1(\cos a), xU_2(\cos a))$. Cos($a$) has to be rational (using the Cos Rule in the triangle shows that.) Say

$\cos(a) = \frac{p}{q}$ with p and q both integers; in order for $(x, xU_1(\cos(a)), xU_2(\cos(a)))$ to all be integers with no common factor, *x* would have to be a perfect *n*th power. 'The proof becomes a five-liner!' I whispered. 'All that parametrisation – a waste of time!'

I wrote to Derek reflectively. 'I take a trig approach and get nowhere, whereupon you point me towards a triple using just geometry, then I use a computer, and we start to spot patterns... and some time later, after quite a lot of sweat on both sides, we get a solution, which then heads us back to a trig-type solution, that is really only a few lines (!) I am not very knowledgeable on poetry, but I like T.S.Eliot's lines;

"We shall not cease from exploration, and the end of our exploring will be to arrive where we started and to know the place for the first time."

I am left a little downcast - it is nice to think one has found something new that is not obvious - but then, maybe that is the fate of all proofs. Maybe Fermat's Last Theorem will be a fiver-liner in a hundred years' time.'

In fact, the 'putting to bed' took a while; my 'fiver-liner' assessment had been premature. It was as if the result did not want to go quietly, an unruly child who was refusing to place his head upon the pillow. I sent Derek the following;

Suppose we take $n = 8$ *and say* $\cos\theta = \frac{19}{30}$ *(wild surmise!) Then what will* $U_8(\frac{19}{30})$ *look like?*

The term at the front will be $a_8\left(\frac{19}{30}\right)^8$. *Certainly multiplying* $U_8(\frac{19}{30})$ *by* 30^8 *will give an integer.*

BUT... $a_8 = 2^8$.

So if we multiply by 30^8, *x, y and z will have a common factor of at least 2.*

So cancel the 2^8, *and multiply instead by* 15^8 *(which is still a perfect 8th power.)*

But then consider the next term in the polynomial, $a_7\left(\frac{19}{30}\right)^7 = a_7\left(\frac{19}{15\times 2}\right)^7$.

So the 15^7 *cancels okay, but we then need to be sure that* 2^7 *divides into* a_7. *And so on.*

But eventually the result succumbed. I spotted a helpful identity in Wikipedia;

$$\mathbf{U}_{n+1}(x) = 2x\mathbf{U}_n(x) - \mathbf{U}_{n-1}(x)$$

Derek provided a lightning argument for this;

$$\begin{aligned}\sin\theta\,[2\cos\theta\,\mathrm{U}_n(\cos\theta) - \mathrm{U}_{n-1}(\cos\theta)] &= 2\cos\theta\sin((n+1)\theta) - \sin(n\theta)\\ &= \sin(n+2)\theta + \sin(n\theta) - \sin(n\theta)\\ &= \sin(n+2)\,\theta.\end{aligned}$$

which allowed him to now finally piece together our definitive Pineapple Triangles proof.

Definition 1

A monic polynomial in x is one whose highest power of x has coefficient 1.

Definition 2

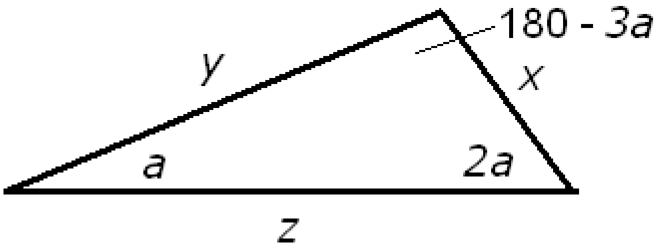
Let $V_n(x) = U_n\left(\frac{x}{2}\right)$.

Lemma

For all n, $V_n(x)$ is a monic polynomial in x with integer coefficients.

Proof of lemma

$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$, so $V_{n+1}(x) = xV_n(x) - V_{n-1}(x)$,
and by induction, we have our result.



$x : y : z = 1 : U_{n-1}(\cos a) : U_n(\cos a)$ (by the Sin Rule.)

Replace cos a with $\frac{2p}{q}$, where p and q have no common factor

(cos a is rational by the Cos Rule.)

So, $(x, y, z) = \left(k: kV_{n-1}\left(\frac{p}{q}\right) : kV_n\left(\frac{p}{q}\right)\right)$,

and, since $V_{n-1}(x)$ and $V_n(x)$ are monic polynomials,

x, y and z are integers with no common factor if, and only if, $k = q^n$.

Our work on parametrisation had, of course, not been a waste of time. It enabled us to produce the integer triples that give pineapple triangles.

2. $(x, y, z) = (a^2, da, d^2 - a^2)$
3. $(x, y, z) = (a^3, d^2a - a^3, d^3 - 2da^2)$
4. $(x, y, z) = (a^4, d^3a - 2da^3, d^4 - 3d^2a^2 + a^4)$
5. $(x, y, z) = (a^5, d^4a - 3d^2a^3 + a^5, d^5 - 4d^3a^2 + 3da^4)$
6. $(x, y, z) = (a^6, d^5a - 4d^3a^3 + 3da^5, d^6 - 5d^4a^2 + 6d^2a^4 - a^6)$
7. $(x, y, z) = (a^7, d^6a - 5d^4a^3 + 6d^2a^5 - a^7, d^7 - 6d^5a^2 + 10d^3a^4 - 4da^6)$
8. $(x, y, z) = (a^8, d^7a - 6d^5a^3 + 10d^3a^5 - 4da^7, d^8 - 7d^6a^2 + 15d^4a^4 - 10d^2a^6 + a^8)$
9. $(x, y, z) = (a^9, d^8a - 7d^6a^3 + 15d^4a^5 - 10d^2a^7 + a^9, d^9 - 8d^7a^2 + 21d^5a^4 - 20d^3a^6 + 5da^8)$

Choosing positive integers a and d so that gcd(a, d) = 1 always gives (x, y, z) to be integer sides of a pineapple triangle with no common factor (as along as x, y and z form a triangle at all), and moreover, Derek's full arguments show that all such pineapple triangles correspond to some such values of a and d.

I meant what I said to Derek about Fermat's Last Theorem – there is an unproved hypothesis called the ABC Conjecture which, if true, reduces the proof of FLT to a few pages. We might begin an educational debate here; should one always give the whole story behind a proof, as I have done in this account? To do so is to ask a lot of one's readers in terms of time – this article is nine pages long, while our final proof is a page. But not to do so drains the life out of the conjecture. Teaching that proceeds 'Theorem-Proof-Examples' is in fact reversing the process of discovery, turning live mathematics into dead mathematics. In our exposition as teachers, we need to find a happy compromise, simplifying helpfully whilst still conveying the thrill of the chase.

Jonny Griffiths, 22 August 2009