A pair of Fibonacci-like polynomials arising from a special mapping

1 Introduction

There are many examples in mathematics of infinite sequences of polynomials defined by simple recurrence relations. Sometimes these sequences arise naturally in pairs. A reasonably well-known example of this is given by the pair comprising the sequences of Chebyshev polynomials of the first and second kinds. The former and latter are denoted by $\{T_n(x)\}$ and $\{U_n(x)\}$, respectively. Chebyshev polynomials of the first kind are defined by way of the following recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

where $T_0(x) = 1$ and $T_1(x) = x$, while those of the second kind are given by

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

with $U_0(x) = 1$ and $U_1(x) = 2x$.

As well as satisfying the same recurrence relation, albeit with different initial conditions, $T_n(x)$ and $U_n(x)$ are related in many other ways [4], and, through their mathematical properties, might be seen very much as a pair.

Likewise, the sequences of Fibonacci and Lucas polynomials, denoted by $\{F_n(x)\}$ and $\{L_n(x)\}$, respectively, may be regarded as a pair. The Fibonacci polynomials are generated by the recurrence relation

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x),$$

where $F_0 = 0$ and $F_1(x) = 1$, while the Lucas polynomials arise from

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x),$$

with $L_0 = 2$ and $L_1(x) = x$. Note that, on setting $x = 1$ in the above recurrences, we obtain the Fibonacci and Lucas sequences, respectively. It may also be checked that $F_{2n}(x) = F_n(x)L_n(x)$. See [5] for further ways in which these sequences are related.

Whilst exploring some aspects of particularly mappings in the plane known as QRT maps, the authors came across a pair of polynomial sequences possessing many similarities to both the pairs discussed above. In this article we define these sequences and give some of their interesting mathematical properties. We note that there are deep mathematical ideas associated with QRT maps, although we shall not be concerning ourselves with these here. Instead, we give just a basic outline in order to help place this article in context.

2 Some brief comments on QRT maps

In this section we provide, for the interested reader, some of the technical background concerning QRT maps. Familiarity with every detail, however, is not essential for an understanding of this article.

We start by considering some biquadratic polynomial $p(x, y)$. By "biquadratic" we mean that $p(x, y)$ is a function in two variables x and y that may be written in either of the forms $a_1(y)x^2 + b_1(y)x + c_1(y)$ or $a_2(x)y^2 + b_2(x)y + c_2(x)$. The biquadratic curve defined by $p(x, y)$ is the set of points (x, y) in the plane for which $p(x, y) = 0$. From a geometrical point of view, a QRT map Q is a map on a biquadratic curve that sends each point on this curve to another by way of what are termed horizontal and vertical switches. It is shown in [1] that all QRT maps are of the form $Q = Q_\beta Q_\alpha$, where

$$Q_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)} \\ y \end{pmatrix} \quad \text{and} \quad Q_\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{g_1(x) - y g_2(x)}{g_2(x) - y g_3(x)} \end{pmatrix}$$

for some polynomials f_k and g_k, $k = 1, 2, 3$, each of degree not exceeding 4. Note that $(Q_\alpha)^2 = (Q_\beta)^2 = I$, that is, Q_α and Q_β are both involutions.

It is possible to show that this mapping on the fixed curve $p(x, y) = 0$ may be extended to a mapping on the plane by way of a one-parameter family of curves $p_c(x, y) = 0$ (where c is the parameter). The equation $p_c(x, y) = 0$ may then be solved for c in order to obtain a rational function $F(x, y)$, which is a ratio of biquadratic polynomials.

An example of such a mapping is the Lyness curve $p_c(x, y) = 0$, where $p_c(x, y) = (x + 1)(y + 1)(x + y + 1) - cxy$ for some parameter c. It may be shown that the resulting QRT map Q is given by $Q_\beta Q_\alpha$, where

$$Q_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{y+1}{x} \\ y \end{pmatrix} \quad \text{and} \quad Q_\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{x+1}{y} \end{pmatrix}.$$

Here, $f_1(y) = y+1$, $g_1(x) = x+1$, $f_2(y) = g_2(x) = 0$ and $f_3(y) = g_3(x) = -1$. We also have that $Q^5 = I$, and so the map Q is periodic; the only periods possible for QRT maps are 2, 3, 4, 5 and 6 [3].

The sequence of iterates $\{Q^k : k \in \mathbb{N}\}$ is known as the *discrete dynamical system* generated by the QRT map Q, where time is indexed by k. As a consequence of its definition, the one-parameter family of biquadratic curves $p_c(x, y) = 0$ is in fact a set of level curves for $F(x, y)$, where

$$F(x, y) = \frac{(x+1)(y+1)(x+y+1)}{xy}.$$

This can be generalised; $F_a(x, y) = \frac{(x+1)(y+1)(x+y+a)}{xy}$ is invariant under the map $(x, y) \mapsto (y, \frac{y+a}{x})$. See [1] for further details on the technical aspects of QRT maps.

In [2] we considered some simple aspects of the behavior of particular instances of these systems as they evolved in discrete time. Here we explore the mathematical properties of a pair of related polynomials that arose on generalising one of these iterated maps.

3 Our mapping and the resultant sequences

The iterated map we consider, $u_n(x, y)$, leads to a sequence of rational functions in x and y by way of the recurrence

$$u_n = \frac{-u_{n-2}u_{n-1} + au_{n-1}^2 + bu_{n-1} + c}{u_{n-1} - u_{n-2}} \tag{1}$$

for $n \geq 2$, where $u_0(x, y) = x$, $u_1(x, y) = y$ and $a, b, c \in \mathbb{R}$. Iteration gives us the following:

$$u_2(x, y) = \frac{-xy + ay^2 + by + c}{y - x},$$

$$u_3(x, y) = \frac{x^2 - (2a+1)xy - bx + (a^2+a)y^2 + (a+1)by + ac}{y - x},$$

$$u_4(x, y) = \frac{p_4x^2 + q_4xy + r_4bx + s_4y^2 + t_4by + v_4c}{y - x},$$

$$u_5(x, y) = \frac{p_5x^2 + q_5xy + r_5bx + s_5y^2 + t_5by + v_5c}{y - x},$$

and so on, where

$$
\begin{aligned}
p_4 &= a, \\
q_4 &= -2a^2 - 2a + 1, \\
r_4 &= -a - 1, \\
s_4 &= a^3 + 2a^2 - 1, \\
t_4 &= a^2 + 2a + 1, \\
v_4 &= a^2 + a,
\end{aligned}
$$

and

$$
\begin{aligned}
p_5 &= a^2 + a, \\
q_5 &= -2a^3 - 4a^2 + 1, \\
r_5 &= -a^2 - 2a - 1, \\
s_5 &= a^4 + 3a^3 + a^2 - 2a, \\
t_5 &= a^3 + 3a^2 + 2a, \\
v_5 &= a^3 + 2a^2 - 1.
\end{aligned}
$$

Note that the maps

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{-xy+ay^2+by+c}{y-x} \\ y \end{pmatrix}$$

and

$$S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{-xy+ax^2+bx+c}{x-y} \end{pmatrix}$$

are both involutions, and so ST is a QRT map, with invariant biquadratic curve

$$x^2 + y^2 + k(x-y) - (1+a)xy - by - c = 0.$$

It turns out, somewhat unexpectedly, that the rational functions $u_n(x,y)$ generated by the recurrence (1) 'retain their shape' in that the denominator remains as $y-x$ while the numerator is always of the form $p_nx^2 + q_nxy + r_ny^2 + s_nbx + t_nby + v_nc$, where each of p_n, q_n, r_n, s_n, t_n and v_n is a polynomial in a for which the degree is a simple function of n. As alluded to above, this behaviour is certainly not typical for expressions generated iteratively by mappings of this type; indeed, the polynomials in both x and

y comprising the numerator and the denominator of the resultant rational functions generally increase rapidly in their degrees as n increases.

Is it possible to find an underlying pattern in the polynomials in a given by p_i, q_i, r_i, s_i, t_i and v_i? Careful examination reveals that there are two sequences of polynomials $\{P_n(a)\}$ and $\{Q_n(a)\}$, called here $\{P_n\}$ and $\{Q_n\}$ for simplicity, each defined recursively by the same recurrence but with differing starting values, such that for $n = 2k$,

$$u_n(x, y) = \frac{(P_{k-1}x - P_k y - bQ_k)(Q_{k-1}x - Q_k y) + cP_{k-1}Q_k}{y - x} \tag{2}$$

and, for $n = 2k + 1$,

$$u_n(x, y) = \frac{(P_{k-1}x - P_k y - bQ_k)(Q_k x - Q_{k+1}y) + cP_k Q_k}{y - x}. \tag{3}$$

The sequence of polynomials $\{P_n(x)\}$ is given by

$$P_{n+1}(a) = (a + 1)P_n(a) - P_{n-1}(a)$$

for $n \geq 0$, where $P_0(a) = 1$ and $P_1(a) = a$, and $\{Q_n(x)\}$ by

$$Q_{n+1}(a) = (a + 1)Q_n(a) - Q_{n-1}(a)$$

for $n \geq 0$, with $Q_0(a) = 0$ and $Q_1(a) = 1$. The first few polynomials in each of these sequences are given in Tables 1 and 2.

n	$P_n(a)$
0	1
1	a
2	$a^2 + a - 1$
3	$a^3 + 2a^2 - a - 1$
4	$a^4 + 3a^3 - 3a$
5	$a^5 + 4a^4 + 2a^3 - 5a^2 - 2a + 1$
6	$a^6 + 5a^5 + 5a^4 - 6a^3 - 7a^2 + 2a + 1$
7	$a^7 + 6a^6 + 9a^5 - 5a^4 - 15a^3 + 5a$
8	$a^8 + 7a^7 + 14a^6 - a^5 - 25a^4 - 9a^3 + 12a^2 + 3a - 1$
9	$a^9 + 8a^8 + 20a^7 + 7a^6 - 35a^5 - 29a^4 + 18a^3 + 15a^2 - 3a - 1$

Table 1: $P_n(a)$ for $n = 0$ to $n = 9$.

n	$Q_n(a)$
0	0
1	1
2	$a+1$
3	a^2+2a
4	a^3+3a^2+a-1
5	$a^4+4a^3+3a^2-2a-1$
6	$a^5+5a^4+6a^3-2a^2-4a$
7	$a^6+6a^5+10a^4-9a^2-2a+1$
8	$a^7+7a^6+15a^5+5a^4-15a^3-9a^2+3a+1$
9	$a^8+8a^7+21a^6+14a^5-20a^4-24a^3+3a^2+6a$

Table 2: $Q_n(a)$ for $n = 0$ to $n = 9$.

4 Some properties of our sequences

There are some simple relationships between the two sequences of polynomials $P_n(a)$ and $Q_n(a)$, the most obvious of which are

(a) $Q_{n+1}(a) - Q_n(a) = P_n(a)$;

(b) $Q_{n+1}(a)P_n(a) - Q_n(a)P_{n+1}(a) = 1$.

These are both simply proved by induction. On substituting (a) into (b) we obtain

$$Q_{n+1}^2(a) - Q_n(a)Q_{n+2}(a) = 1,$$

which contrasts with the following identity for the Fibonacci polynomials;

$$F_{n+1}^2(a) - F_n(a)F_{n+2}(a) = (-1)^n.$$

Returning to (2) and (3), it is now a straightforward (if lengthy) task to prove these results using the identities above; it helps also to have the explicit forms for $P_n(a)$ and $Q_n(a)$, which are

$$P_n(a) = \frac{a-1+x}{2x}\left(\frac{a+1+x}{2}\right)^n + \frac{x+1-a}{2x}\left(\frac{a+1-x}{2}\right)^n,$$

while

$$Q_n(a) = \frac{1}{x}\left(\frac{a+1+x}{2}\right)^n - \frac{1}{x}\left(\frac{a+1-x}{2}\right)^n,$$

where x is $\sqrt{(a+3)(a-1)}$.

It is also the case that $P_n(2) = F_{2n+1}$ and $Q_n(2) = F_{2n}$, where F_n denotes the nth Fibonacci number. Both of these results may easily proved by induction, on noting that, when specialising to $n = 2$, the polynomial recurrence relation for $\{P_n(x)\}$ is given by

$$P_{n+1}(2) = 3P_n(2) - P_{n-1}(2)$$

for $n \geq 0$, where $P_0(2) = 1$ and $P_1(2) = 2$, and that for $\{Q_n(x)\}$ becomes

$$Q_{n+1}(2) = 3Q_n(2) - Q_{n-1}(2)$$

for $n \geq 0$, with $Q_0(2) = 0$ and $Q_1(2) = 1$. It may also be noted that $P_n(1) = 1$ and $Q_n(1) = n$ for all $n \geq 0$.

For ease of notation in what follows, we once again take the liberty in this proof of using P_n and Q_n to denote the polynomials $Q_n(a)$ and $P_n(a)$, respectively. On examining the Tables 1 and 2, it would seem that P_n is divisible by a if, and only if, $n \equiv 1 \mod 3$, whereas Q_n is divisible by a if, and only if, $3|n$. Let us prove that this is true.

Theorem 4.1. *$a|P_n$ if, and only if, $n \equiv 1 \mod 3$, while $a|Q_n$ if, and only if, $3|n$.*

Proof. Let us start be considering the polynomials P_n. Note first that P_1 is divisible by a, but P_0 is not. Now assume that the statement concerning P_n is true for all n satisfying $0 \leq n \leq 3k-2$, where $k \geq 1$.

Using the inductive hypothesis and the definition of P_n,

$$\begin{aligned} P_{3k-1} &= (a+1)P_{3k-2} - P_{3k-3} \\ &= ia(a+1) - j \end{aligned}$$

for some integers i and j such that j is not divisible by a. This shows that a is not a factor of P_{3k-1}.

We also have

$$\begin{aligned} P_{3k} &= (a+1)P_{3k-1} - P_{3k-2} \\ &= i(a+1) - ja \end{aligned}$$

for some i and j where i is not divisible by a, showing that P_{3k} is not divisible by a.

Finally,

$$\begin{aligned} P_{3k+1} &= (a+1)P_{3k} - P_{3k-1} \\ &= (a+1)\left[(a+1)P_{3k-1} - P_{3k-2}\right] - P_{3k-1} \\ &= P_{3k-1}\left[(a+1)^2 - 1\right] - (a+1)P_{3k-2} \\ &= a(a+2)P_{3k-1} - (a+1)P_{3k-2}, \end{aligned}$$

which divisible by a since P_{3k-2} is. This completes the proof of the result for P_n. The proof of the corresponding divisibility result for Q_n follows along similar lines. □

n	$Q_n(a)$
0	0
1	1
2	$a+1$
3	$a(a+2)$
4	$(a+1)(a^2+2a-1)$
5	$(a^2+a-1)(a^2+3a+1)$
6	$a(a+1)(a+2)(a^2+2a-2)$
7	$(a^3+2a^2-a-1)(a^3+4a^2+3a-1)$
8	$(a+1)(a^2+2a-1)(a^4+4a^3+2a^2-4a-1)$
9	$a(a+2)(a^3+3a^2-1)(a^3+3a^2-3)$

Table 3: $Q_n(a)$ factorised for $n = 0$ to $n = 9$.

On considering Table 3 we might note that the polynomials $Q_n(a)$ would appear to possess a less obvious divisibility property, namely that if $m|n$ then $Q_m(a)|Q_n(a)$. We now prove that this is indeed the case (and thus the polynomials $\{Q_n(a)\}$ form a *divisibility sequence*).

Theorem 4.2. *If $m|n$ then $Q_m(a)|Q_n(a)$.*

Proof. We first show, by induction on n, that

$$Q_{m+n} = Q_{m+1}Q_n - Q_mQ_{n-1}. \tag{4}$$

for any $m \in \mathbb{N}$. (Note that for the Fibonacci numbers,

$$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}.)$$

When $n = 1$ the right-hand side is

$$Q_{m+1}Q_1 - Q_m Q_0 = Q_{m+1},$$

which is equal to the left-hand side in this case. Let us now assume that (4) is true for $n = k$, where $k \in \mathbb{N}$. Then, by the inductive hypothesis and the definition of Q_n, we have

$$\begin{aligned}
Q_{m+(k+1)} &= Q_{(m+1)+k} \\
&= Q_{m+2}Q_k - Q_{m+1}Q_{k-1} \\
&= Q_k\big[(a+1)Q_{m+1} - Q_m\big] - Q_{m+1}\big[(a+1)Q_k - Q_{k+1}\big] \\
&= (a+1)Q_{m+1}Q_k - Q_m Q_k - (a+1)Q_{m+1}Q_k + Q_{m+1}Q_{k+1} \\
&= Q_{m+1}Q_{k+1} - Q_m Q_k \\
&= Q_{m+1}Q_{k+1} - Q_m Q_{(k+1)-1},
\end{aligned}$$

giving the desired result.

We may now go on to complete the proof of the theorem, utilising induction once more. By way of initialisation, we have $Q_m | Q_m$ for any $m \in \mathbb{N}$. Assume now that $Q_m | Q_{mn}$ for some $n = k \in \mathbb{N}$. Using (4), we obtain

$$\begin{aligned}
Q_{m(k+1)} &= Q_{mk+m} \\
&= Q_{mk+1}Q_m - Q_{mk}Q_{m-1}.
\end{aligned}$$

Since $Q_m | Q_m$ and, by way of the inductive hypothesis, $Q_m | Q_{mk}$, it follows that

$$Q_m \big| \big(Q_{mk+1}Q_m - Q_{mk}Q_{-1}\big),$$

thereby completing the proof of the theorem. □

Furthermore, it is reasonably straightforward to show that the ordinary generating functions, $G(a, x)$ and $H(a, x)$, for the sequences $\{P_n(a)\}$ and $\{Q_n(a)\}$, respectively, are given by

$$G(a, x) = \frac{1 - x}{1 - (a+1)x + x^2}$$

and

$$H(a, x) = \frac{x}{1 - (a+1)x + x^2}.$$

These generating functions allow us easily to calculate certain infinite sums involving either $\{P_n(a)\}$ or $\{Q_n(a)\}$. For example, on noting that the following sum converges, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{P_n\left(\frac{1}{2}\right)}{2^n} &= G\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \\ &= \frac{1-\frac{1}{2}}{1-\frac{3}{4}+\frac{1}{4}} \\ &= 1.\end{aligned}$$

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