<u>Finding Patterns in Simultaneous Equations</u>

Ever more powerful calculators on the market, that'll differentiate, interpolate, and make you a cup of tea. An unequivocal boon, or a way to avoid acquiring knowledge you should have at your fingertips? It is often said in their favour that these new machines will enable us and our students to ask investigational questions about tougher topics - freed of the drudgery of endless mechanical processes, we can spot patterns that would not be accessible otherwise. This is an account of how I and a few of my students tried this, with simultaneous equations.

Simultaneous-equation-solving is not an activity I particularly enjoy, and I'm quite happy to leave it to my calculator, especially if I'm doing twenty in a row. I started with two unknowns.

$$\mathbf{ax + by = c}$$

$$\mathbf{dx + ey = f}$$

The obvious thing seemed to be to tap in **1, 2, 3, 4, 5, 6** for the coefficients.

This solves to **x = -1, y = 2.**

Trying **2, 3, 4, 5, 6, 7** gave the same solution.

Trying my first set of numbers backwards? **6, 5, 4, 3, 2, 1** gave the same solution again.

How about numbers that form part of an arithmetic series?

8, 11, 14, 17, 20, 23, and those same numbers in reverse gave the same solution.

I was wondering by now what I had to do NOT to get this solution, or if my calculator had packed up. I decided to use a little algebra.

$$\mathbf{ax + (a + d)x = (a + 2d)}$$

$$\mathbf{(a + 3d)x + (a + 4d)y = (a + 5d)}$$

Everything cancelled remarkably obligingly to give **x = -1, y = 2.**

I wondered how to generate some more thoughts on proof from this situation.

Clearly, from the above, if we have numbers from an Arithmetic Series, the solution is **(-1, 2).**

Does a solution of **(-1,2)** mean that the numbers are from an Arithmetic Series?

A counter-example here is very easy to construct: pick any four numbers for **a, b, d, e,** not in any pattern, substitute in **(-1,2)** and get values for **c** and **f.**

If **A** is the statement: *the coefficients are from an arithmetic series*

and **B** is the statement: *the solution is* **(-1,2)**

then seeing that **A** implies **B** but that **B** does not imply **A** would be a good experience for my students, I felt.

By now the questions were multiplying, and I took the whole situation along to the regular lunchtime group. Almost immediately, suggestions were made.

Sam said: "Think of a number, take some other number away from this, this add the second number you thought of on to the original. This gives you your top three numbers. Do the same with a different pair of numbers to get the bottom three numbers."

For example, $\mathbf{12x + 7y = 17}$

and $\mathbf{11x + 8y = 14}$

give the solution **(2,-1)**

"You always get (**2, -1**) ".

Nice to get away from **(-1, 2),** if not very far away. In fact, swapping the xs and ys takes us back to arithmetic series, and gives you the old solution. Sam's idea does suggest, however, that a triplet from an arithmetic series followed be a triplet from another arithmetic series gives **(-1, 2),** which it does. Indeed, going back to ideas of proof, we can now show that if **A** is the statement *the coefficients are two triplets from arithmetic series*, and **B** is the statement *the solution is **(-1, 2),*** then **A** is true if and only if **B** is true.

Now we turned to the Fibonnacci sequence, and the session really took off.

Take any six consecutive numbers from the Fibonnacci Sequence starting **1, 1, 2, 3, 5 ...** gave the solution **(1,1),** which was fairly obvious when we thought about it. Using these numbers backwards gave **(1, -1).**

Taking every second number from the sequence, starting anywhere, gave **(-1,3).**
Every third number gave **(1,4).**
Every fourth number gave **(-1,7).**

Discoveries were coming so thick and fast now that two or three people were shouting out simultaneously (perhaps appropriate in view of the exercise.)

“The value for x swaps around...”
“The value for y is Fibonnacci too..”

We tried every fifth in the sequence, and every sixth to verify.

One fascination for me was how easy it seemed to be to get whole numbers as a solution. It raised for me the following question:

Pick six integers a, b, c, d, e, f at random from 1 to 100 (with replacement).

Solve $$\mathbf{ax + by = c}$$

$$\mathbf{dx + ey = f}$$

What is the probability that you get an integer solution for x and y?

A graphics calculator program to look at this experimentally suggested the answer would be somewhere between 1/400 and 1/500.

What happens when you leap to equations in three unknowns?

Trying **1, 1, 2, 3, 5** ....did not produce an answer within the machine's capabilities.

How about **1, 1, 1, 3, 5, 9, ...,** where the next term is found by adding the previous three? Entering the first twelve terms of the sequence gave the solution **(1,1,1).** What happens if we enter every other number from the sequence? This gave the solution **(1,1,3).** The limits imposed by the machine (only taking six-digit numbers) were beginning to pinch. What if we take the first twelve numbers and making every other number negative? We got the solution **(-1, 1, -1).** Trying to predict the result of each trial was a good test of our intuition.

So, this all leaves a large number of loose ends, and a multitude of directions to explore, from matrices, Fibonnacci properties, to the meaning of proof, to algebraic manipulation. I was left feeling that this is a fertile situation, made possible by the calculators. You can imagine similar investigations with more advanced machines over differentiating functions that lie in patterns, or factorising polynomials that lie in patterns. My students were able to take off on this exercise. They found this to be mathematics that was fun, perhaps because we were quite likely to be asking questions that not many other people had asked before.

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