Square Tessellations

Twenty-five years ago I embarked upon a university course called Vision and Structure, a cross-curricular mathematics-art study where tessellations played a large part. My collection of university notes has dwindled since leaving, until now I possess only a few remaining pages, all of which come from that one unit. Perhaps they represent the gold left after years of panning: whenever I chance across a tiling now, I can feel a distant part of me waking up. The other day, I chanced across the tessellation shown in Figure 1:

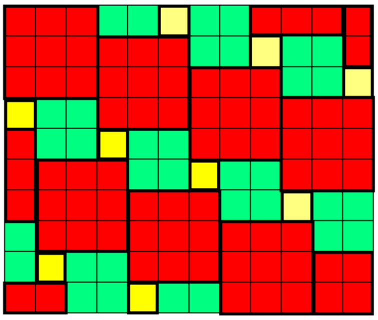

Fig. 1

This can be seen as a tessellation of the tile in Figure 2.

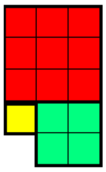

Fig. 2

This tile is built from a 1-square, a 2-square and a 3-square (throughout this article an 'n-square' means a square of side n units.) Thus the tessellation in Figure 1 contains equal numbers of 1-squares, 2-squares and 3-squares. Now I knew that a tile built from a 1-square and a 2-square can tessellate (see Figure 3) - there are many patios up and down the land that testify to that, mine included.

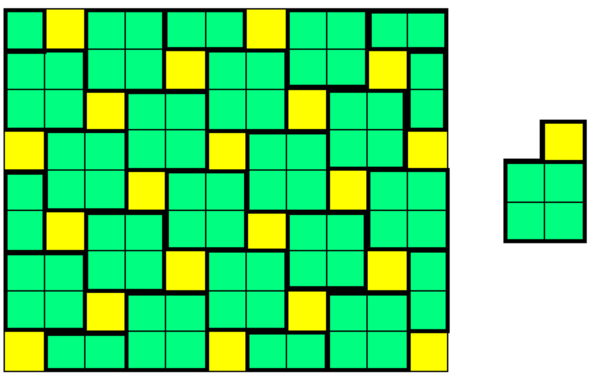

Fig. 3

The question arises, can you always find a tessellation constructed from a tile containing equal numbers of 1-squares, 2-squares, ... up to n-squares, for any natural number n?

Let's start by looking at n = 4. We can certainly arrange our four tiles into what we might call an L-shape tile, shown in Figure 4:

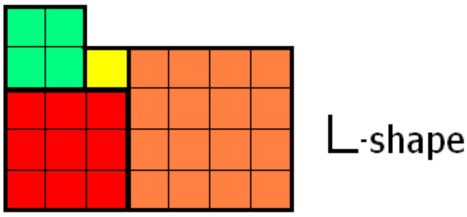

Fig. 4

This certainly tessellates (see Figure 5):

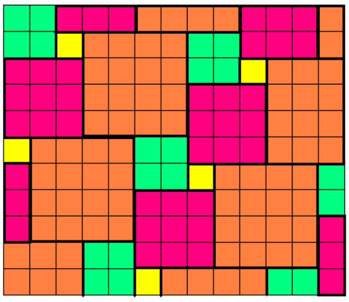

Fig. 5

Indeed, it is easy to convince yourself that every L-shape tile will tessellate in this way. So how far can we go with this? 1, 2, 3, 4 and 5-squares form the L-shape tile shown in Figure 6.

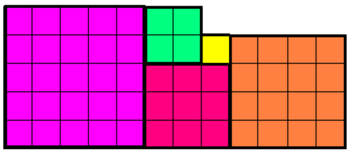

Fig. 6

However, try as we might, we cannot form an L-shape tile from a 1-square, a 2-square… up to a 6-square (have a go!) But we can get what we might call an S-shape tile from these squares (see Figure 7):

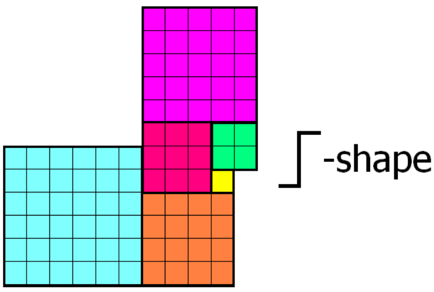

Fig. 7

Does this tessellate? It does, forming the attractive tessellation in Figure 8.

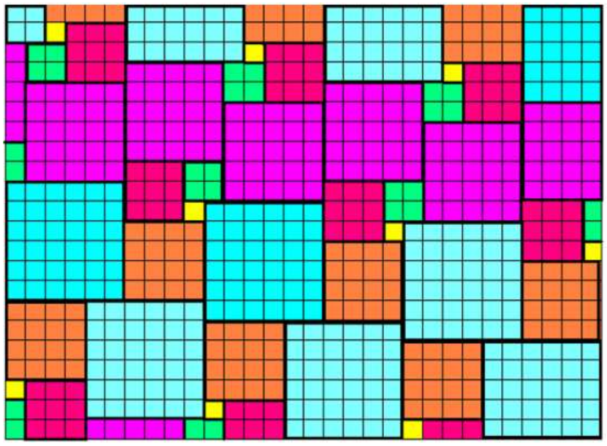

Fig. 8

We can add a 7-square to get another S-shape (Figure 9), which also tessellates (Figure 10).

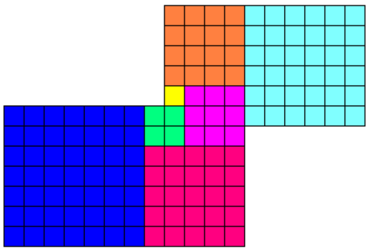

Fig. 9

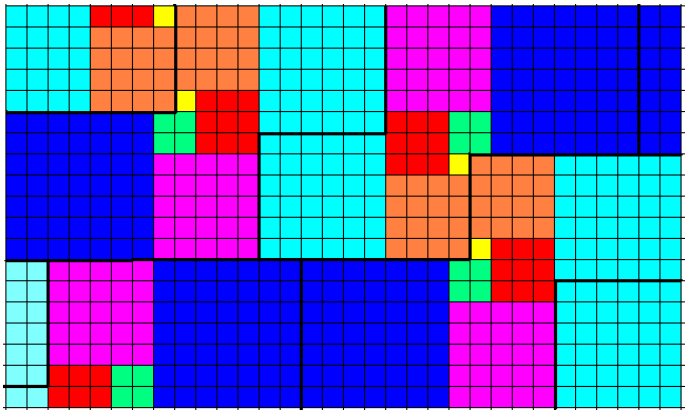

Fig. 10

We might be forgiven for thinking that all S-shapes tessellate, but this is untrue, as experimenting with the tile in Figure 11 shows:

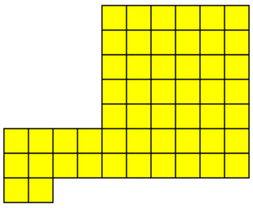

Fig. 11

At this point we seem to come to the end of the road. However we play around with the tiles, we can't seem to put a 1-square through to an 8-square together in a way that will create an L-shape, or indeed a helpful S-shape. Perhaps a different tack is needed.

Suppose that S_n is the statement: "There exists a rectangle made up of equal numbers of 1-squares through to n-squares." If S_n is true for all n, then as a rectangle clearly tessellates, we will have a tile that meets our requirements.

S_1 is clearly true, as is S_2 (Figures 12 and 13).

Fig. 12

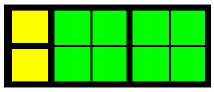

Fig. 13

If we now add two 3-squares to Figure 13, we can make two rectangles that will not fit together to make a third.

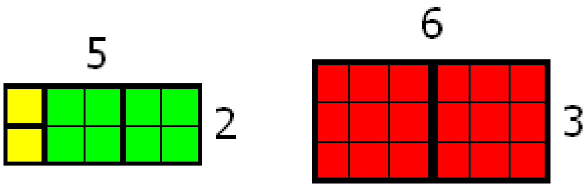

Fig. 14

However, if we take 2×3 = 6 copies of Figure 14, we can then build a single rectangle from the result.

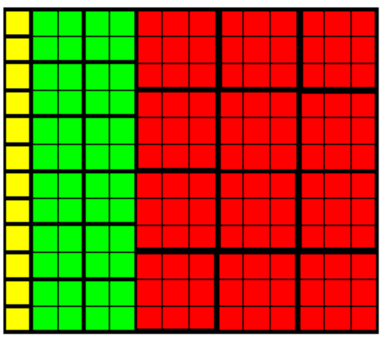

Fig. 15

So S_3 is true. Can we carry out this procedure in the general case? An argument using induction shows that we can.

Suppose S_n is true and there is a rectangular tile (called A) containing equal numbers of 1-squares through to n-squares (suppose also there are k of each, and that A is p by q). We can make k (n+1)-squares into a k(n+1) by (n+1) rectangle (call this B).

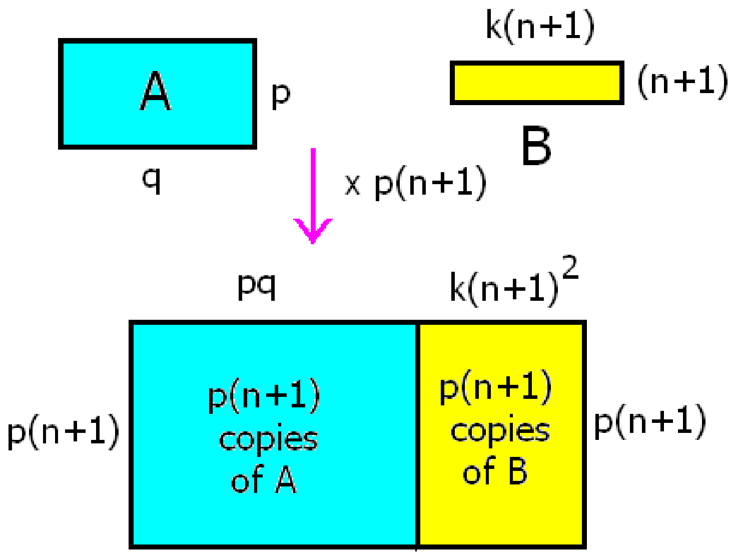

Fig. 16

Now take (n+1)p copies of the diagram (we may not need as many as this.) We can combine the copies of A into a (n+1)p by pq rectangle, while the copies of B will form an (n+1)p by $k(n+1)^2$ rectangle. These may be simply combined into a single rectangle, containing equal numbers of 1 through to (n+1)-squares (there will be kp(n+1) of each.)

So if S_n is true, then S_{n+1} is true, and by induction, a tile that tessellates made from equal numbers of 1-squares through to n-squares is possible for all n. Maybe a challenge if you happen to need a new patio?

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