Sumlines

This simple question occurred to me the other day: what happens if you take three non-parallel non-vertical lines in the form $y = mx + c$ and 'add' them? For example:

$$y = 2x - 1,\ y = 3x + 2 \text{ and } y = x + 2 \text{ add to } 3y = 6x + 3, \text{ or } y = 2x + 1.$$

A name was needed, and naturally enough I called the sum of the lines 'the sumline.' Plotting all four lines, I found the sumline crossed the triangle formed by the original three lines. Would this always happen? I mentioned this problem to the Editor, who came up with this elegant proof.

Firstly, note that the sumline is well-behaved under translation. Suppose we carry out the transformation:

$$x' = x + a,\ y' = y + b.$$

Then the trio of lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ (which have sumline $y = \frac{m_1 + m_2 + m_3}{3}x + \frac{c_1 + c_2 + c_3}{3}$) become:

$$y' - b = m_1(x' - a) + c_1,\ y' - b = m_2(x' - a) + c_2 \text{ and } y' - b = m_3(x' - a) + c_3,$$

and these have sumline $y' - b = \frac{m_1 + m_2 + m_3}{3}(x' - a) + \frac{c_1 + c_2 + c_3}{3}$, which is simply the translation applied to the original sumline.

If the three lines are concurrent, then clearly their sumline must pass through the point of intersection. In all other cases, the three lines will form a triangle. Let us say that this triangle has vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3), where $x_1 < x_2 < x_3$ (no side can be vertical). Now translate the axes to make (x_2, y_2) the origin. Thus the equations of the sides become (without loss of generality):

$$y = px,\ y = qx \text{ and } y = rx + s.$$

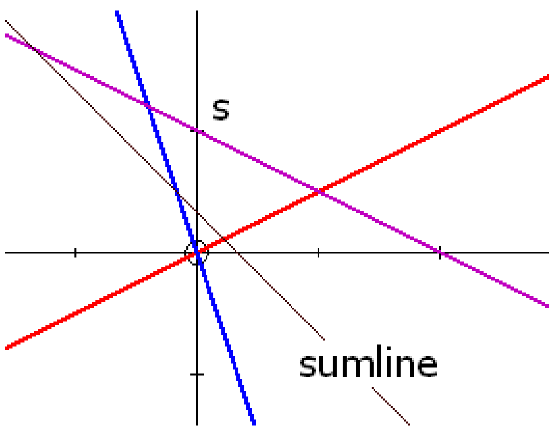

Fig. 1

Thus the sumline is $y = \frac{p+q+r}{3}x + \frac{s}{3}$, and it is clear by considering the y-intercept of this line that it must cross the triangle.

Do the lines have to be in the form y = mx + c? For example, what happens if you add three lines as follows:

$2x + 3y + 5 = 0$, $x + 6y - 5 = 0$ and $-2x + y + 1 = 0$, adding to $x + 10y + 1 = 0$?

We are in effect taking a weighted average of the lines this time. Will this sumline always cross the triangle as before? The Editor came to my aid again, this time with a counterexample:

$9y + x = 0$, $y + 9x = 0$ and $-9y - 5x - 9 = 0$ give the sumline $y + 5x - 9 = 0$,

which does not cross the triangle.

However, if we write our lines in the form y = mx + c, and if we weight our lines with positive constants, then the resulting sumline does cross the triangle. As before, translate the origin of coordinates to a vertex so that the y-axis crosses the triangle.

$$k_1y = m_1k_1x,\ \ k_2y = m_2k_2x \ \text{ and } k_3y = m_3k_3x + c_3k_3 \text{ add to}$$
$$y = \frac{m_1k_1 + m_2k_2 + m_3k_3}{k_1 + k_2 + k_3}x + c_3\frac{k_3}{k_1 + k_2 + k_3}.$$

The point $(0, c_3\frac{k_3}{k_1 + k_2 + k_3})$ is on the line and inside the triangle, since $\frac{k_3}{k_1 + k_2 + k_3} < 1$.

The next natural question: will four lines that define a quadrilateral have a sumline that crosses it? The answer is, it depends what you mean by a quadrilateral. Consider Fig. 2:

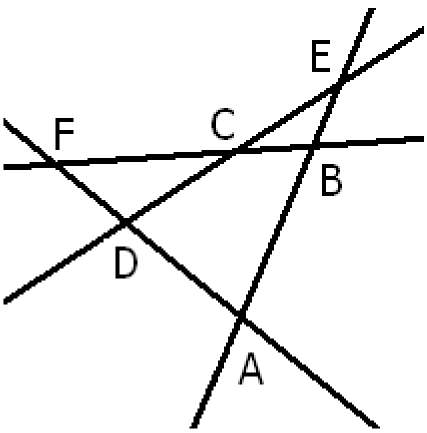

Fig. 2

ABCD is what we might call the 'conventional quadrilateral' formed by the four lines, while the figure ADFCEB is called the 'complete quadrilateral'. The sumline of the four lines that border a conventional quadrilateral need not cross it. Consider Figure 3:

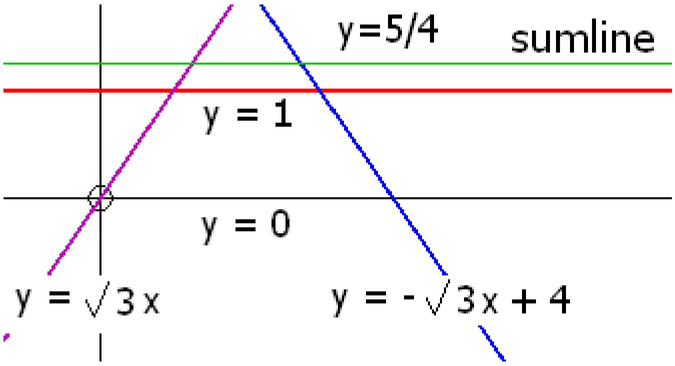

Fig. 3

However, the sumline will cross the complete quadrilateral formed by the four lines. There is a neat if informal argument that shows this. First, consider the sumline of two non-parallel non-vertical lines. Clearly their sumline will go through their point of intersection. Moreover, the sumline is "vertical-averse", that is, it will 'avoid' the regions containing the vertical line through the point of intersection. This is because it must have a gradient that lies between the gradients of the two lines, which means it must lie in the grey zones in Fig. 4.

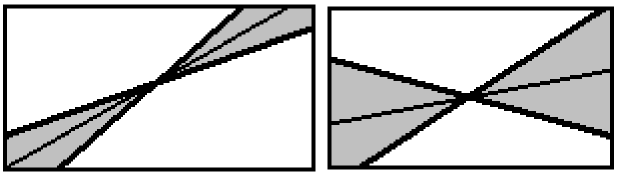

Fig. 4

The next thing to note is that the sumline of four lines is the sumline of (the sumline of two of the lines) and (the sumline of the other two).

Now the complete quadrilateral will be made up of a conventional quadrilateral DEFG with two adjacent triangles AEF and BGF, as in Fig. 5.

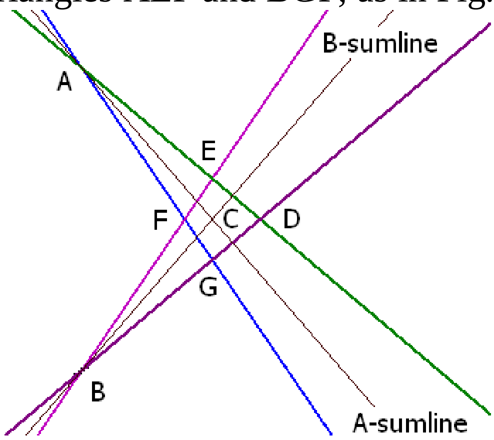

Fig. 5

Define the A-sumline as the sumline of the two lines through A. In Figure 5 the A-sumline crosses the B-sumline within the conventional quadrilateral at C, so the sumline of all four lines must go through C and thus cross the complete quadrilateral.

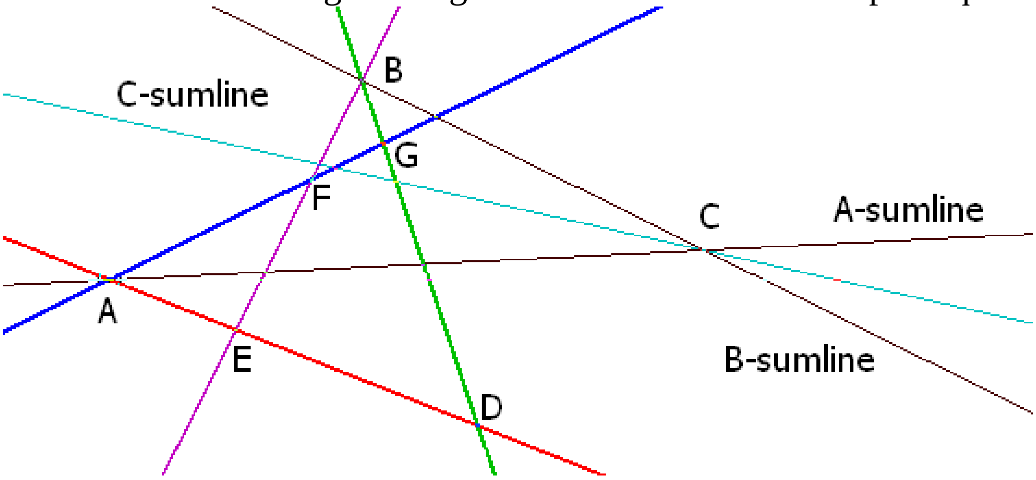

Fig.6

In Figure 6, the A-sumline and the B-sumline meet outside the conventional quadrilateral at C, but the C-sumline (the sumline of all four lines) must now have a gradient between that of AC and BC, and so will cross the complete quadrilateral. It is easy to satisfy oneself that this will happen for all other cases, including those in which there are parallel sides.

There are some conventional quadrilaterals that will always be crossed by their sumline, however. Take any parallelogram. Translate it so that its centre is at the origin. The four sides must now be of the form $y = m_1x \pm c_1$ and $y = m_2x \pm c_2$, and so their sum will be a line through the origin. This simple argument extends to any 2n-sided shape that has opposite sides equal and parallel. (Does this kind of shape have a name? I am tempted to call any 2n-sided shape with opposite sides parallel a ‘paragon’ and if in addition the opposite sides are equal, to call this a 'paragon of virtue' (!)) This will of course include all regular 2n-agons. Indeed, one conjecture is that the sumline will always cross a general paragon: it will surely cross a hexagonal paragon, as Figure 7 shows.

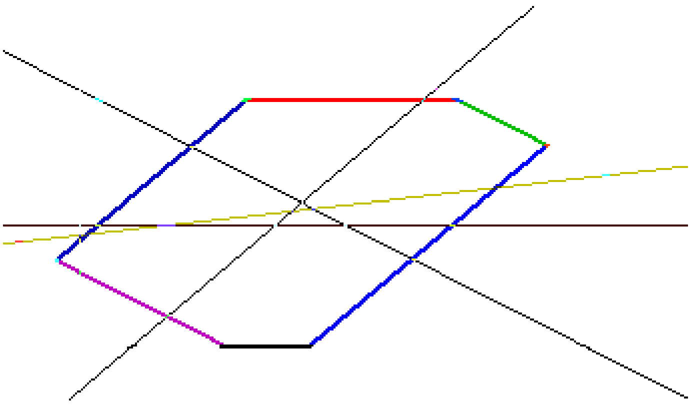

Fig. 7

Adding the parallel lines in pairs gives three sumlines that cross in a triangle within the hexagonal paragon. The sumline of these three sumlines is the sumline of the whole figure, and it must cross this triangle, by our earlier work.

So what happens for regular (2n+1)-agons? Take the equilateral triangle: surely the sumline passes through its centre? Perhaps surprisingly, the answer is no. Take the triangle whose sides are formed by $y = 0$, $y = \sqrt{3}x$ and $y = -\sqrt{3}x + 2\sqrt{3}$. The sumline is $y = \frac{2\sqrt{3}}{3}$, which does not pass through the triangle's centre at $(1, \frac{\sqrt{3}}{3})$.

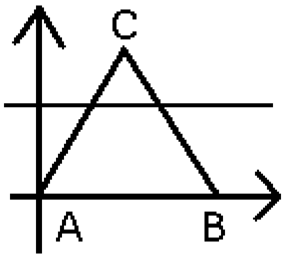

Fig. 8

I might point out at this point that a triangle's sumline is not well-behaved under rotation, that is, the sumline of a rotated triangle is not the sumline of the original when rotated. Take the triangle in Fig. 8 and rotate it about A so that BC and then CA are horizontal.

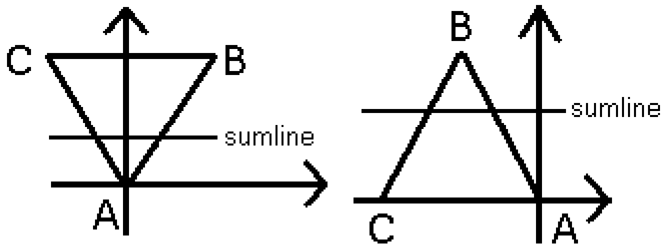

Fig. 9

However, a sumline is well-behaved under enlargement. Given two similar shapes oriented in the same way, translate them so that the enlargement becomes one centred at the origin, as in Figure 10.

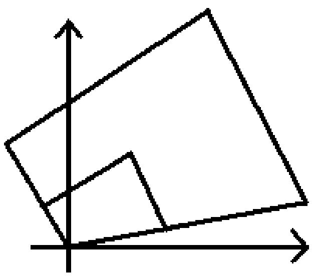

Fig. 10

Clearly all gradients are unaffected by the enlargement, and the y-intercepts are simply multiplied by the scale factor of the enlargement, so the sumline of the enlarged shape is in the same proportion to its host as the original sumline is to the original shape.

So a triangle has no centre through which its sumlines pass. However, Fig. 8 and Fig. 9 suggest an alternative conjecture. All three sumlines shown are tangents to the equilateral triangle's incircle. Could this be true however the equilateral triangle is aligned?

This conjecture is not true for triangles in general. Consider the 3-4-5 triangle in Figure 11 (which has an incircle radius 1):

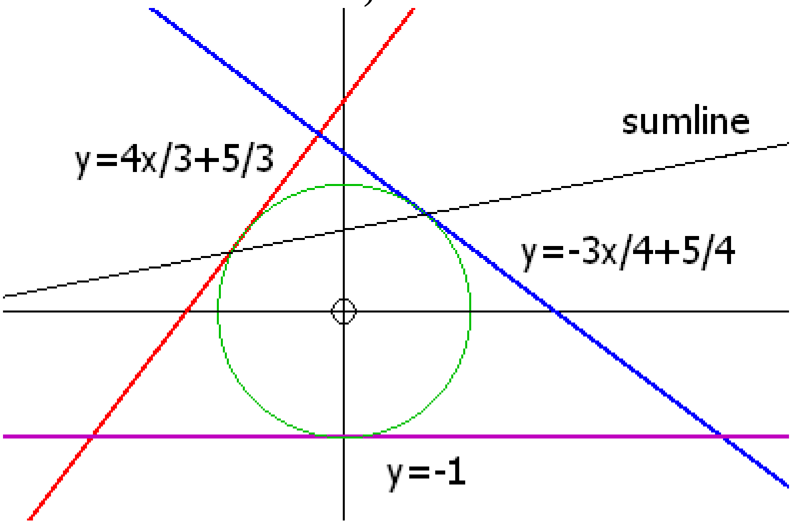

Fig. 11

The sumline will be y = 7x/36 + 23/36, which crosses the incircle (radius 1, centre the origin) in two places. (Please note, the sumline here does <u>not</u> go through the points where the sides touch the incircle!) However, might our conjecture remain true for equilateral triangles?

Take two points on the unit circle centre the origin, and call them (cos θ, sin θ) and (cos Φ, sin Φ). The straight line joining them has equation:

$$y = \frac{\sin\theta - \sin\phi}{\cos\theta - \cos\phi} x + \sin\theta - \cos\theta \, \frac{\sin\theta - \sin\phi}{\cos\theta - \cos\phi},$$ which simplifies to:

$$y = \frac{-1}{\tan(\frac{\theta+\phi}{2})} x + \frac{\cos(\frac{\theta-\phi}{2})}{\sin(\frac{\theta+\phi}{2})}.$$

So let us take three points on the circumference of this circle that are the vertices of an equilateral triangle, say (cos θ, sin θ), (cos (θ + 120), sin (θ + 120)) and (cos (θ + 240), sin (θ + 240)). The sumline of the three lines joining them must be:

$$3y = -\left(\frac{1}{\tan(\theta+60)}+\frac{1}{\tan(\theta+180)}+\frac{1}{\tan(\theta+300)}\right)x$$
$$+\frac{1}{2}\left(\frac{1}{\sin(\theta+60)}+\frac{1}{\sin(\theta+180)}+\frac{1}{\sin(\theta+300)}\right)$$

After simplification using some standard trigonometry, this becomes:

$$y = \cot\theta\cot(\theta-60)\cot(\theta+60)\,x + \frac{1}{8}\operatorname{cosec}\theta\operatorname{cosec}(\theta-60)\operatorname{cosec}(\theta+60).$$

Let us call this $y = Px + Q$. The incircle is centred at the origin, radius $\frac{1}{2}$. So where does the sumline cross this circle, that is, $x^2 + y^2 = \frac{1}{4}$? Solving the two equations simultaneously:

$$x^2 + (Px+Q)^2 = \frac{1}{4} \quad x^2(1+P^2) + x(2PQ) + Q^2 - \frac{1}{4} = 0$$

Thus the sumline will be a tangent to the circle if and only if "$b^2 = 4ac$", that is,

$4P^2Q^2 = 4(1+P^2)(Q^2 - \frac{1}{4})$, which gives $4Q^2 = 1 + P^2$.

So we need:

$$\frac{1}{16}\operatorname{cosec}^2\theta\operatorname{cosec}^2(\theta-60)\operatorname{cosec}^2(\theta+60) = 1 + \cot^2\theta\cot^2(\theta-60)\cot^2(\theta+60),$$

or $\frac{1}{16} = \sin^2\theta\sin^2(\theta-60)\sin^2(\theta+60) + \cos^2\theta\cos^2(\theta-60)\cos^2(\theta+60)$.

Happily, it is easy to show by expanding using standard trigonometry that this is indeed an identity. So the sumline will always touch the incircle of the equilateral triangle, whatever θ may be.

How to extend this result? The obvious thought is that the sumline might always touch the incircle of a regular (2n+1)-sided polygon. I have tested this for a regular pentagon and a regular heptagon, and it would appear to be true. The trigonometry for 5, 7, 9… sides would seem to extend naturally from the case for the equilateral triangle.

If true, this is quite a result. Given a regular 1000-agon, we know the sumline will go through its centre. Tweak this slightly into a regular 1001-agon, and the sumline still crosses the figure, but only just, traversing the tiny annulus between the shape's inscribed and circumscribed circles. Now if the figure is changed slightly once more into a regular 1002-agon, the sumline jumps back to passing through the centre again. Adding one side can make a big difference, it seems!

This notion of adding the equations of several curves to generate a sum-curve can be taken further. What happens if you add the equations of three circles to get a sumcircle? If the three circles overlap, it is clear that the sumcircle must enclose the common area, because:

$(x-a_1)^2 + (y-b_1)^2 < r_1^2$ and $(x-a_2)^2 + (y-b_2)^2 < r_2^2$ and $(x-a_3)^2 + (y-b_3)^2 < r_3^2$

$(x-a_1)^2 + (y-b_1)^2 + (x-a_2)^2 + (y-b_2)^2 + (x-a_3)^2 + (y-b_3)^2 < r_1^2 + r_2^2 + r_3^2.$

It is fun to put the equations of the three circles and the sumcircle into a graphing program like Autograph and watch the way the sumcircle surrounds the intersection as the parameters are varied. Sometimes the most enjoyable results are the simple ones!

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